Numerical Evaluation of Dynamic Response by Using Modified Newmark's Method

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ABSTRACT

It is well known that the Newmark's method is considered one of the most popular methods for structural dynamic analysis. In this study, starting from the basic Newmark's method, a new accurate method is investigated and developed. The basic idea of the proposed method is to use Richardson's extrapolation to improve the basic Newmark's method. To observe the accuracy of the proposed method, several numerical tests are performed for a single degree-of-freedom (SDM) dynamic system and the results are compared with results from Newmark's method and the exact solution. The results show that the proposed method improves the solution accuracy of the structural dynamic problems compared to the Newmark's method. Moreover, the results of the free oscillating case show that the modified Newmark's method has more computational efficiency compared to the Newmark's method.

KEYWORDS: Newmark's method, Richardson's extrapolation, Structural dynamic analysis, Numerical analysis, Modified Newmark's method, Integration algorithm.

INTRODUCTION

It is usually not possible to find an analytical solution of the equation of motion for a structural system if the system is non-linear or the excitation force varies arbitrarily with time. Such problems can be solved by direct numerical integration of the dynamic equilibrium equations. The ease of implementation of direct integration methods has tended to enhance rapidly the popularity of these approaches (Owren et al., 1995; Kim et al., 1997; Laier, 2000; Williamson et al., 2002; Chen et al., 2008; Chang, 2015). In direct integration, the governing equation of motion for the structural system is integrated using a numerical step-by-step procedure (Bathe et al., 1976). Basically, the direct integration method is divided into two general classes; explicit methods and implicit methods. In the explicit methods, the solution at time step $t+\Delta t$ is obtained by considering the equilibrium condition of equation of motion at time t (Subbaraj, 1989a). On the other hand, in the implicit methods, the equation of displacement at the current time step involves the velocity and acceleration at the current time step itself (Subbaraj, 1989b). The most popular explicit methods are: second-order central difference methods (Krieg, 1973), Runge-Kutta methods (Kutta, 1901), stiffly stable methods, predictor-corrector methods, Taylor series schemes, dynamic relaxation method (Namadchi et al., 2016) and the KR-α method (Kolay and Ricles, 2014, 2016, 2017). Likewise, the most popular implicit methods are Newmark's family methods (Newmark, 1959), Wilson-θ (Wilson et al., 1973) and Houbolt methods (Houbolt, 1950). Surveys of both classes of direct time integration

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methods can be found in papers by Subbaraj (1989 a,b). These methods have different characteristics of accuracy and stability and each one has its specialty for a specific problem (Bathe, 1996; Bathe, 2012). However, for general purposes, Newmark's method is one of the most popular methods for earthquake response analysis. In this study, Newmark's method based on Richardson's extrapolation is developed. The key idea of using Richardson's extrapolation in Newmark's method is to minimize the numerical error.

THEORY

Newmark's Beta Method

In 1959, N. M. Newmark developed a whole series of time-stepping solution methods (Newmark, 1959). In this method, the acceleration is assumed to vary in a specific manner over the time step. The finite difference relationships for the Newmark-β method are:

$$\dot{u}_{i+1} = \dot{u}_i + \left[(1-\gamma)\ \Delta t \right] \ddot{u}_i + (\gamma\ \Delta t)\ddot{u}_{i+1} \tag{1}$$

$$u_{i+1} = u_i + (\Delta t)\ \dot{u}_i + \left[(0.5-\beta)(\Delta t)^2 \right] \ddot{u}_i + \left[\beta\ (\Delta t)^2 \right] \ddot{u}_{i+1} \tag{2}$$

where:

$\dot{u}_{i+1}$: Velocity at time step "*i+1*";
$\dot{u}_i$: Velocity at time step " *i* ";
$\ddot{u}_{i+1}$: Acceleration at time step "*i+1*";
$\ddot{u}_i$: Acceleration at time step "*I* ";
Δt: Time step.

The parameter β controls the manner in which the acceleration varies over the time step. For β = 0, the acceleration is assumed to remain constant over the time interval. β = 0.25 corresponds to the assumption of constant average acceleration and β = 1/6 corresponds to the assumption of linear variation of acceleration. Newmark-β method is stable if (Chopra, 2013):

$$\frac{\Delta t}{T_n} \le \frac{1}{\pi\sqrt{2}} \frac{1}{\sqrt{\gamma - 2\beta}} \tag{3}$$

where:

T_n : Natural period.

For γ = 1/2 and β = ¼, this condition becomes:

$$\frac{\Delta t}{T_n} < \infty \tag{4}$$

For γ = 1/2 and β = 1/6, "Eq. (3)" indicates that the linear acceleration method is stable if:

$$\frac{\Delta t}{T_n} \le 0.551 \tag{5}$$

Newmark's Method: Linear System

The time stepping solution using Newmark's method can be summarized in the following algorithm (Bathe, 1996; Chopra, 2013):

1. Initial calculations

1.1 $$\ddot{u}_O = \frac{P_O - c\dot{u}_O - k\,u_O}{m} \tag{6}$$

1.2 $$\hat{k} = k + \frac{\gamma}{\beta(\Delta t)} c + \frac{1}{(\Delta t)^2} m \tag{7}$$

1.3 $$a = \frac{1}{\beta \Delta t} m + \frac{\gamma}{\beta} c \text{; and } b = \frac{1}{2\beta} m + \Delta t \left(\frac{\gamma}{2\beta} - 1 \right) c \tag{8}$$

where: m, c and k are mass, viscous damping coefficient and stiffness of the system, respectively.

2. Calculation for each time step, i

2.1 $$\Delta \hat{P}_i = \Delta P_i + a\,\dot{u}_i + b\,\ddot{u}_i \tag{9}$$

2.2 $$\Delta u_i = \frac{\Delta \hat{p}_i}{k} \tag{10}$$

2.3 $\Delta \dot{u}_i = \frac{\gamma}{\beta \Delta t}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i$ (11)

2.4 $\Delta \ddot{u}_i = \frac{\gamma}{\beta(\Delta t)^2}\Delta u_i - \frac{1}{\beta \Delta t}\dot{u}_i + \left(\frac{1}{2\beta}\right)\ddot{u}_i$ (12)

2.5 $u_{i+1} = u_i + \Delta u_i$, $\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i$, $\ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_i$ (13)

3. *Repetition for the next time step*. Replace *i* by *i+1* and implement steps 2.1 to 2.5 for the next time step.

Richardson's Extrapolation

This method is very powerful in improving the accuracy of an integration scheme (Chapra et al., 1998). It is a procedure which combines several approximations of a certain quantity *I* in such a way to yield a more accurate approximation of *I*. It can be expressed in a general form as follows:

$$I_{j,k} = \frac{4^{k-1} \times I_{j+1,k-1} - I_{j+1,k-1}}{4^{k-1} - 1} \quad (14)$$

Newmark's Method Based on Richardson's Extrapolation

The basic idea is to use the Richardson's extrapolation to improve the Newmark's method. The first step in developing the Newmark's method is to determine successive estimates of the response at a given time by using Newmark's method by progressively doubling the number of sub-intervals. These estimated values can be assigned to Eq. (14). The time stepping solution using Newmark's method based on Richardson's extrapolation for linear structural systems can be summarized in the following algorithm.

1- Calculate $u_{i+1,T}$, $\dot{u}_{i+1,T}$, $\ddot{u}_{i+1,T}$ by using time step $\Delta t = T$ by using Newmark's method.

2- Calculate $u_{i+1,T/2}$, $\dot{u}_{i+1,T/2}$, $\ddot{u}_{i+1,T/2}$ by using time step $\Delta t = \frac{T}{2}$ by using Newmark's method.

3- Calculate $u_{i+1,T/4}$, $\dot{u}_{i+1,T/4}$, $\ddot{u}_{i+1,T/4}$ by using time step $\Delta t = \frac{T}{4}$ by using Newmark's method.

4- Calculate $u_{i+1,l}$, $\dot{u}_{i+1,l}$, $\ddot{u}_{i+1,l}$ by using Richardson's extrapolation by using the following equations:

$$u_{i+1,l} = \frac{16\times\left(\frac{4}{3}\left(u_{i+1,T/4}\right) - \frac{1}{3}\left(u_{i+1,T/2}\right)\right) - \left(\frac{4}{3}\left(u_{i+1,T/2}\right) - \frac{1}{3}\left(u_{i+1,T}\right)\right)}{15}$$

$$\dot{u}_{i+1,l} = \frac{16\times\left(\frac{4}{3}\left(\dot{u}_{i+1,T/4}\right) - \frac{1}{3}\left(\dot{u}_{i+1,T/2}\right)\right) - \left(\frac{4}{3}\left(\dot{u}_{i+1,T/2}\right) - \frac{1}{3}\left(\dot{u}_{i+1,T}\right)\right)}{15}$$

$$\ddot{u}_{i+1,l} = \frac{16\times\left(\frac{4}{3}\left(\ddot{u}_{i+1,T/4}\right) - \frac{1}{3}\left(\ddot{u}_{i+1,T/2}\right)\right) - \left(\frac{4}{3}\left(\ddot{u}_{i+1,T/2}\right) - \frac{1}{3}\left(\ddot{u}_{i+1,T}\right)\right)}{15} \quad (15)$$

5- Repeating for next iteration. Replace *i +1* by *i+2* and repeat calculation steps 1 to 4 for the next time step. This procedure extends also to non-linear systems.

EVALUATION OF THE PROPOSED METHOD

To observe the accuracy of the present integration method, several numerical examples are solved and the predicted solutions are compared to those of Newmark's method and the exact solution. Very important aspects in the evaluation of any numerical time integration methods are: convergence, stability and accuracy. In this study, the proposed integration method is unconditionally stable, because it is based on constant average acceleration method. Accuracy and convergence can be evaluated by examining period elongation and amplitude decay. The percentage error of period elongation can be defined by the difference of the numerical period T_{num} and the exact period T as:

$$PE = \frac{T_{num} - T}{T} \times 100 \qquad (16)$$

The displacement error e_{disp} is the cumulative difference of numerically calculated displacement $u_{num}(t)$ and exact solution $u(t)$ as:

$$e_{disp} = \left| u_{num}(t) - u(t) \right| \qquad (17)$$

Numerical Example

Harmonic Vibration with Viscous Damping

Newmark's method based on Richardson's extrapolation is applied on an SDM system with zero initial displacement and velocity, that is subjected to $P(t)$ defined by a half sine pulse force ($P_O \sin \Omega t$). The equation of motion is:

$$m\ddot{u} + c\dot{u} + k u = P(t) \qquad (18)$$

The response of the system is evaluated by exact solution, Newmark's method and Newmark's method based on Richardson's extrapolation. It is evaluated for different values of system period (T_n), damping ratio (ζ) and time stepping (Δt). Table 1 shows all cases of SDF systems evaluated.

Table 1. Cases of system the dynamic response of which is evaluated

Case	Natural Period (T_n)	Damping Ratio (ζ)	Time Stepping (Δ t)	R=Δ t / T_n
Case 1	0.1	0.05	0.005, 0.01, 0.02, 0.05	0.05, 0.1, 0.2, 0.5
Case 2	0.25	0.05	0.005, 0.01, 0.02, 0.05	0.02, 0.04, 0.08, 0.2
Case 3	0.25	0.8	0.005, 0.01, 0.02, 0.05	0.02, 0.04, 0.08, 0.2
Case 4	2	0.05	0.005, 0.01, 0.02, 0.05, 0.1	0.0025, 0.005, 0.01, 0.025, 0.05
Case 5	2	0.8	0.005, 0.01, 0.02, 0.05, 0.1	0.0025, 0.005, 0.01, 0.025, 0.05
Case 6	3	0.05	0.01, 0.02, 0.05, 0.1, 0.15	0.0033, 0.0167, 0.033, 0.1, 0.05

The exact solution of the above system would be in the form of:

$$u(t) = e^{-\varsigma\,\omega\,t}(A\cos\omega_D t + B\sin\omega_D t) + C\sin\Omega t + D\cos\Omega t$$

$$t \le t_O$$

$$u(t) = e^{-\varsigma\,\omega\,t}\begin{bmatrix} u(0)\cos\omega_d t + \\ \dfrac{\dot{u}(0) + \varsigma\omega\, u(0)}{\omega_D}\sin\omega_D t \end{bmatrix}$$

$$t > t_O \qquad (19)$$

where A, B, C and D are real-valued constants determined by standard procedures.

The predicted solutions of the modified Newmark's method are compared to those of Newmark's method and the exact solution. As seen in Eq. (15), the modified Newmark's method with time step $\Delta t=T$ relies on employing the Newmark's method with three different time steps $\Delta t=T$, $T/2$ and $T/4$. In order to observe the efficiency of the modified Newmark's method, the predicted solution of the present method with the time step $\Delta t=T$ is compared to Newmark's solution with time step $\Delta t=T/4$. Comparison of the results in Figures 1 and 2 shows the difference in results obtained by the use of Newmark's method with time step $\Delta t=T/4$ and the modified Newmark's method with time step $\Delta t=T/4$. These results imply that: a) Both Newmark's method and modified Newmark's method are of superior accuracy

for small time steps (i.e., $\frac{\Delta t}{T_n} \le 0.025$); b) Modified Newmark's method shows a good improvement in the solution accuracy although the used time step is larger

than that used in Newmark's method for $\frac{\Delta t}{T_n} \le 0.075$; c) Damping ratio of the system does not affect the trend of the results of both methods; and d) Modified Newmark's

method with large time step (i.e., $\frac{\Delta t}{T_n} \ge 0.08$) shows less solution accuracy when compared with Newmark's method. Hence, the improvement of solution accuracy of the proposed method is limited to the use of a small time step.

Free Oscillating Case

Free vibration motion with no damping is predicted in this case. The equation of motion is as follows:

$$m\ddot{u} + k\,u = 0 \tag{20}$$

with the initial conditions $u(t) = u_O = 3$ m/sec and $\dot{u}(t) = \dot{u}_O = 0$.

The exact solution for the above system would be:

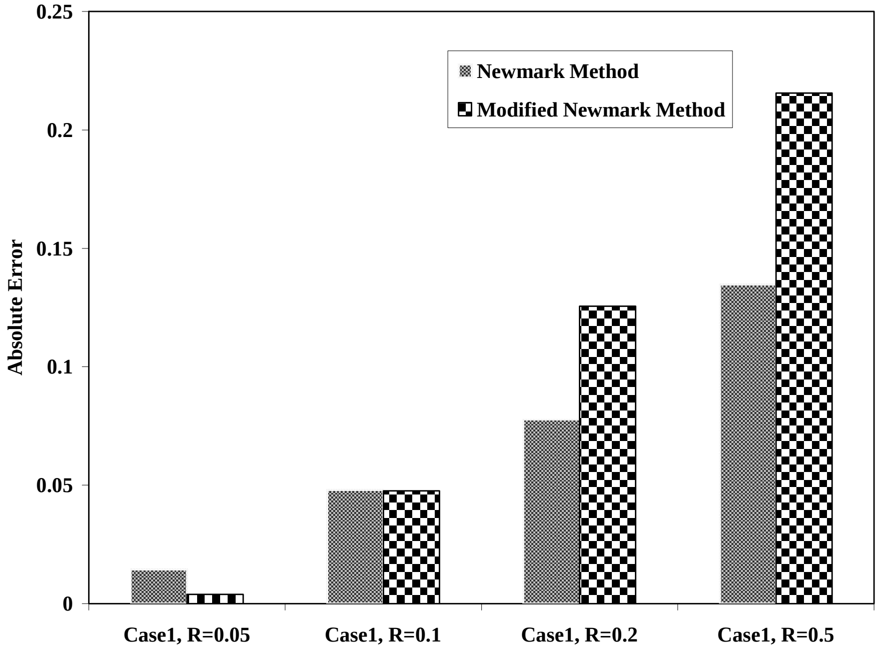
$$u(t) = u_O \cos(\omega t) + \frac{v_O \sin(\omega t)}{w} \tag{21}$$

Different cases of the above system with different values of system period T_n and time stepping are simulated and the predicted results are compared with the exact solution. The simulated results show that the present method predicts the period of the dynamic system accurately as shown in Fig. 3.

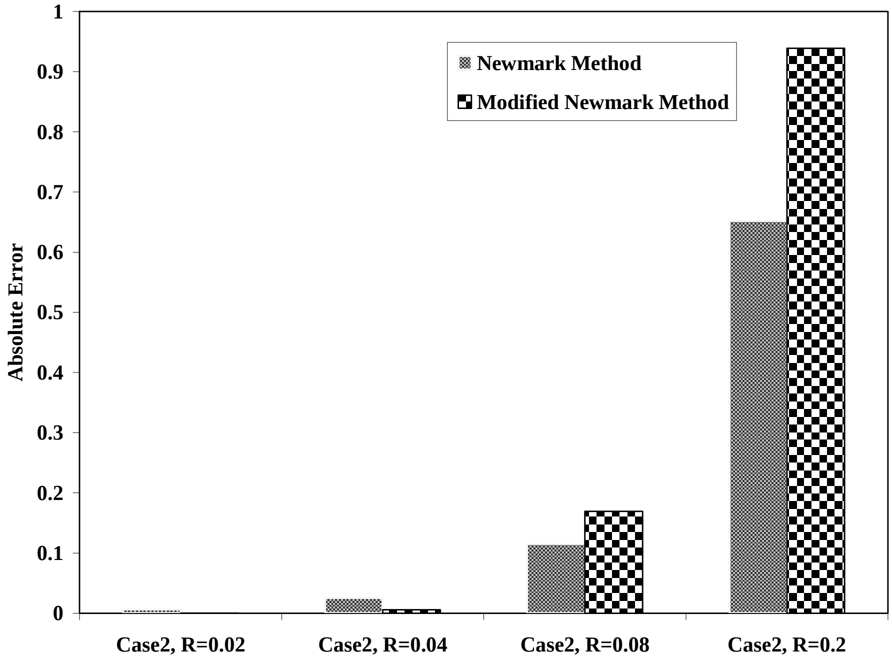
The computational efficiency is also an essential aspect in the evaluation of any time integration method. Modified Newmark's method with time step Δt=T will demand less computational time when compared with Newmark's solution with time step Δt=T/8. To observe the computational efficiency of the present integration method, the predicted solution of the present method with time step Δt=T is compared to Newmark's solution with time step Δt=T/8 in the previous two numerical examples. The results of harmonic vibration with viscous damping show that there is no clear trend that confirms the higher computational efficiency of the present method than that of Newmark's method. But, the results of the free oscillating case show that the modified Newmark's method has more computational efficiency and accuracy than Newmark's method.

CONCLUSION

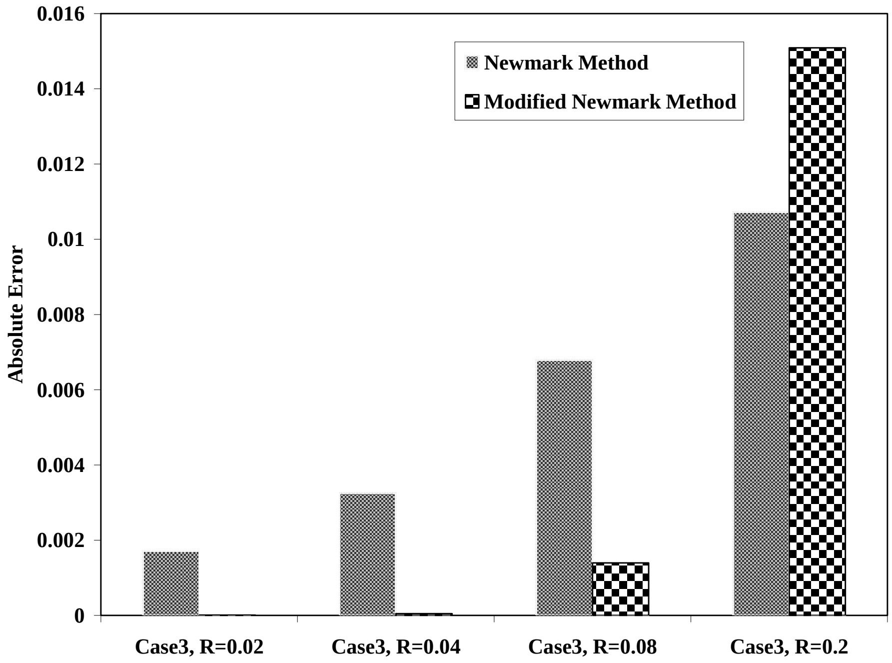
A new step-by-step integration algorithm for structural dynamic systems is presented. The proposed method is a modification of the basic Newmark's method. The basic idea of the proposed method is to use Richardson's extrapolation to improve the basic Newmark's method. Several numerical tests were carried out for an SDF dynamic system. The results confirm that the proposed method has a relative period error that is smaller than that of Newmark's methods. Also, the results support that the proposed method possesses improved accuracy and can be an alternative for solving structural dynamic problems.



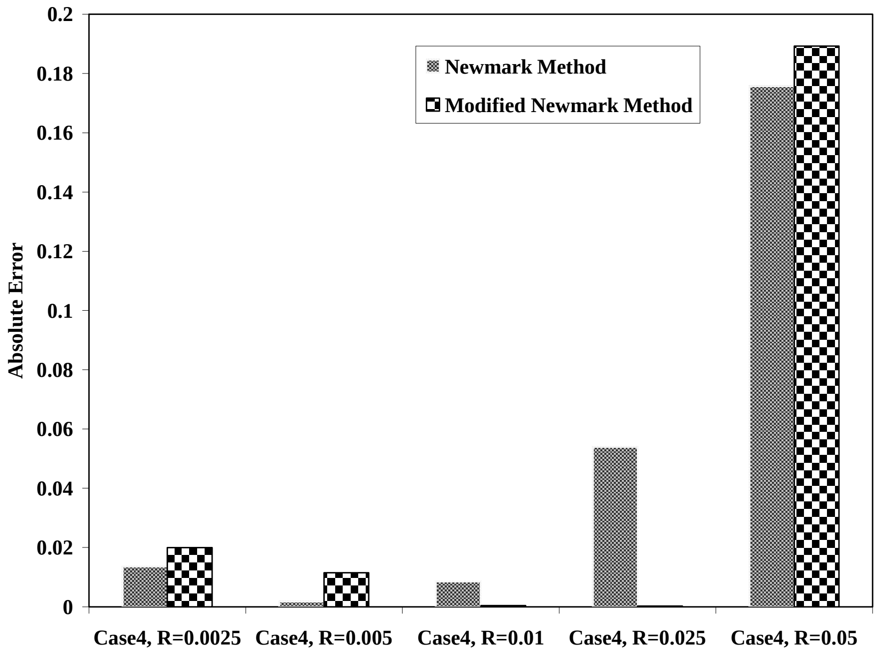
(a) **Case 1**



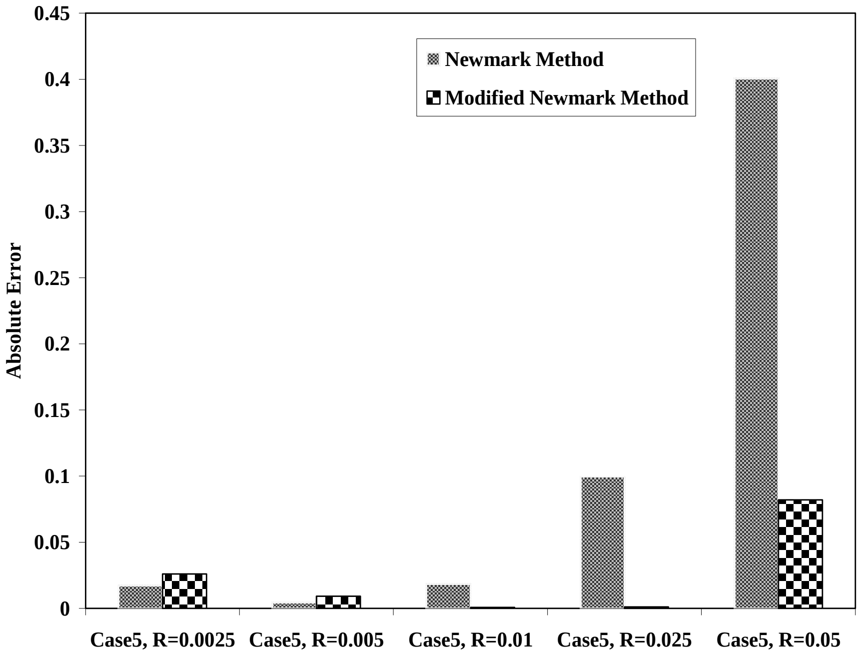
(b) **Case 2**



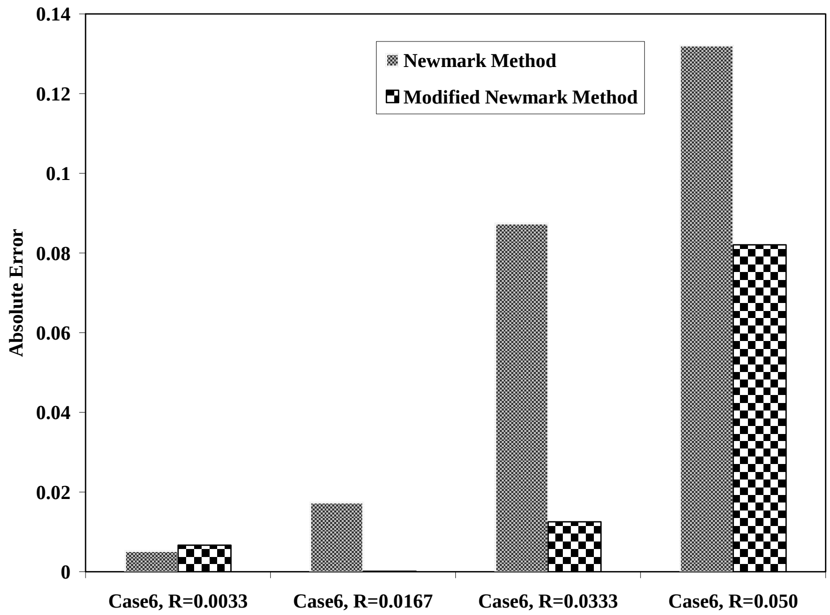
(c) **Case 3**



(d) **Case 4**

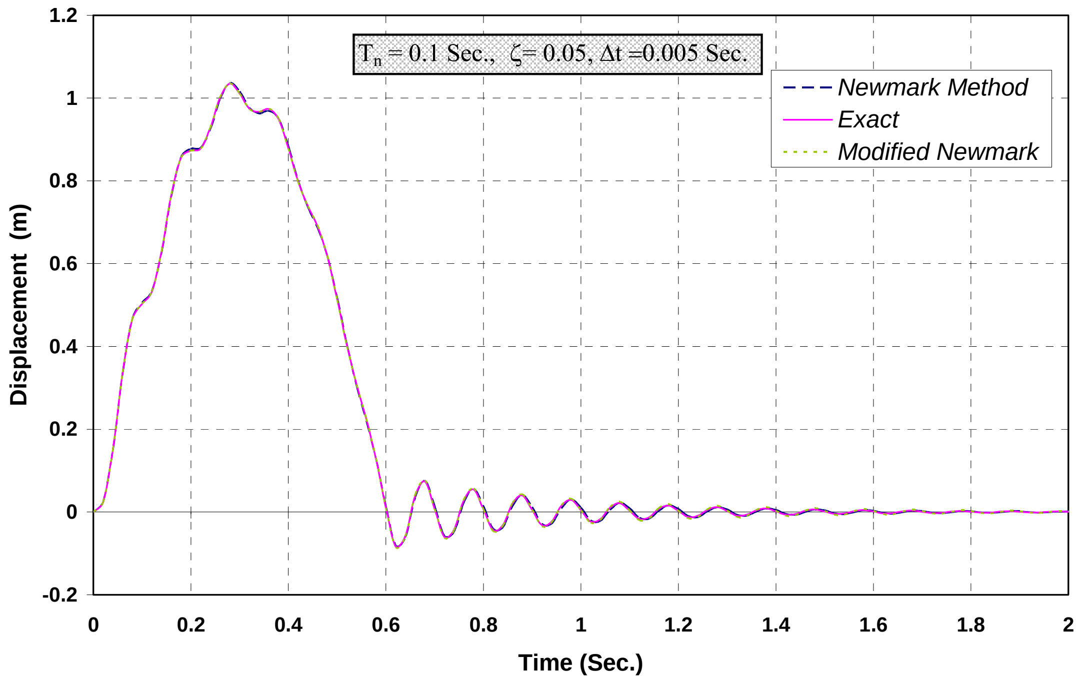


(e) **Case 5**



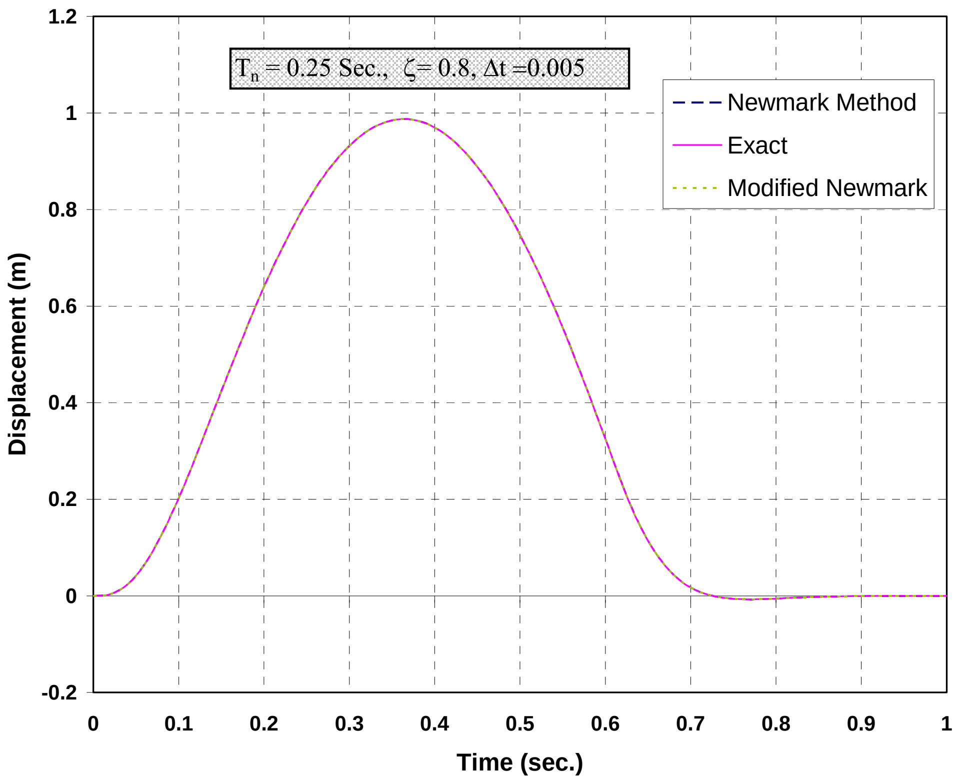
(f) **Case 6**

Figure (1): Absolute displacement error of the proposed method compared with the Newmark's method, where R= $\Delta t/T_n$



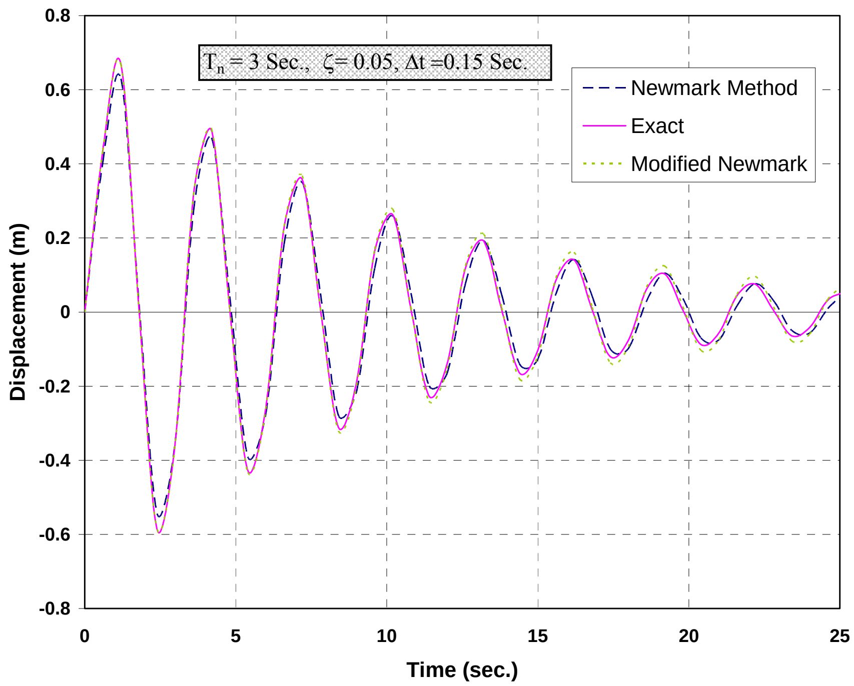
T_n = 0.1 Sec., ζ= 0.05, Δt =0.005 Sec.
Newmark Method
Exact
Modified Newmark
Displacement (m)
Time (Sec.)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

(a)



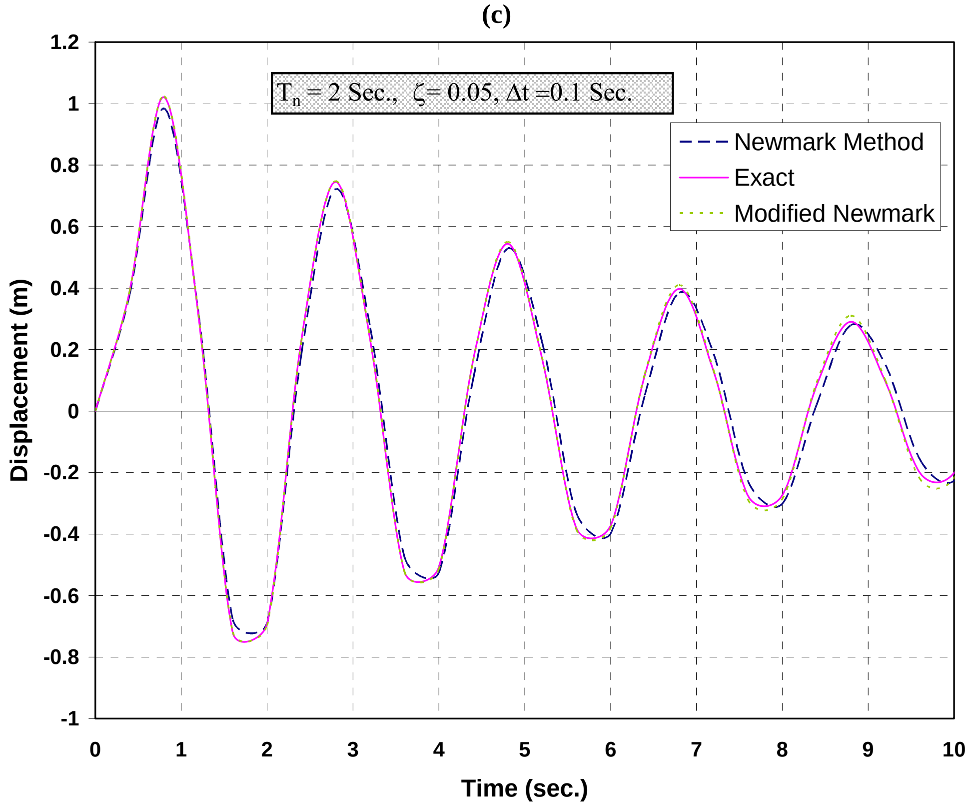
T_n = 0.25 Sec., ζ= 0.8, Δt =0.005
Newmark Method
Exact
Modified Newmark
Displacement (m)
Time (sec.)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

(b)



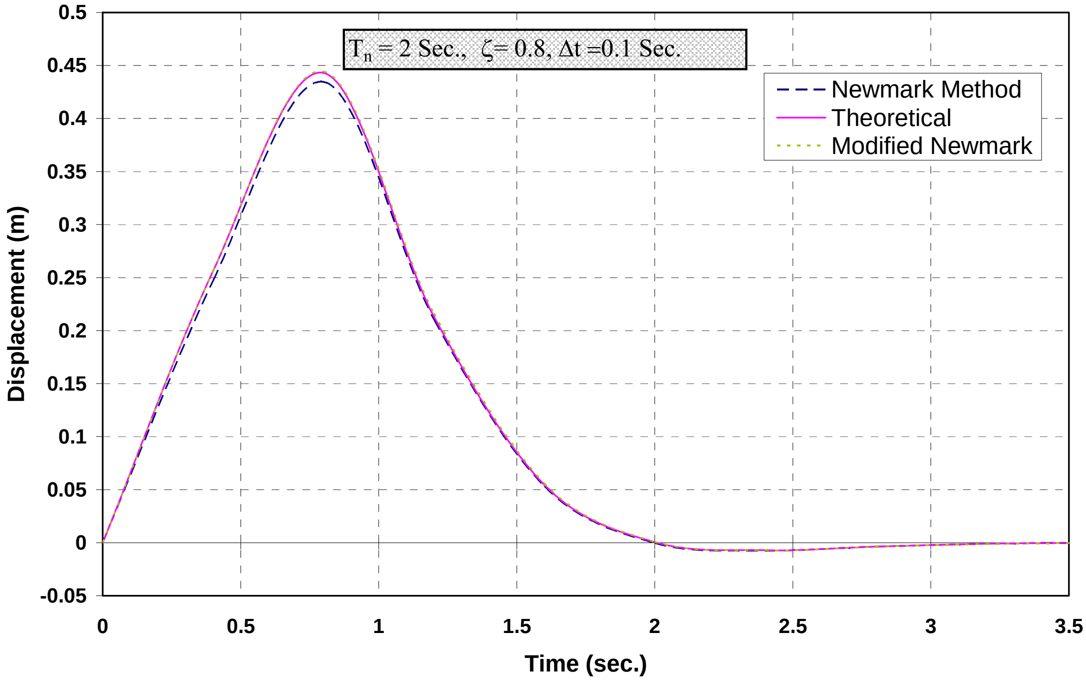
T_n = 3 Sec., ζ= 0.05, Δt =0.15 Sec.
Newmark Method
Exact
Modified Newmark
Displacement (m)
Time (sec.)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25

(c)

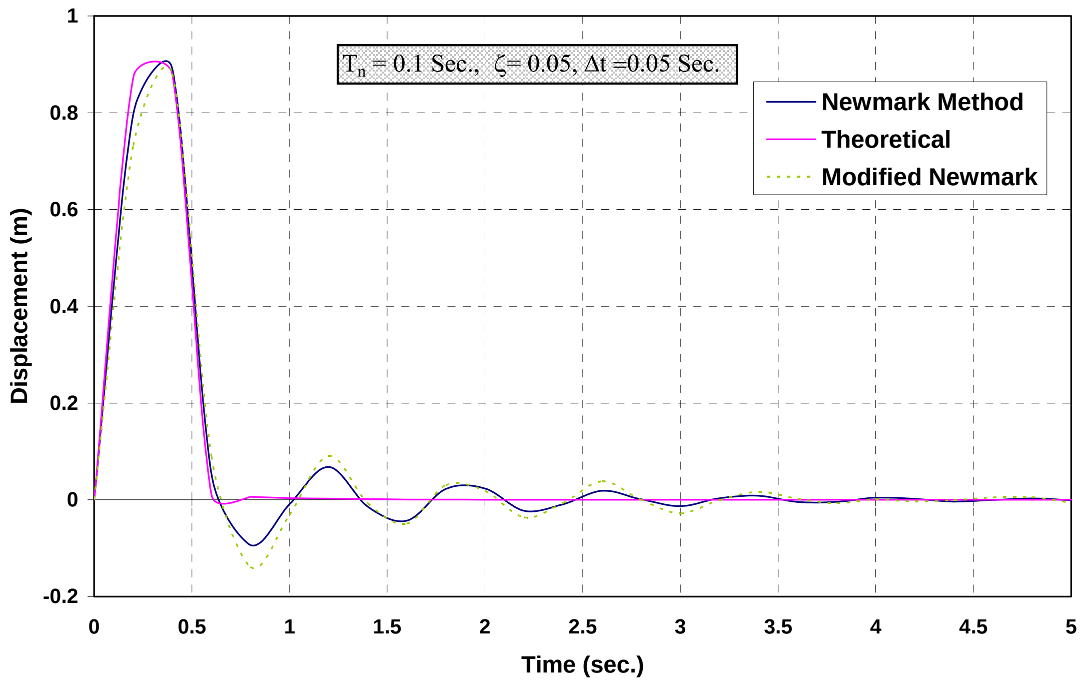


T_n = 2 Sec., ζ= 0.05, Δt =0.1 Sec.
Newmark Method
Exact
Modified Newmark
Displacement (m)
Time (sec.)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
1
2
3
4
5
6
7
8
9
10

(d)

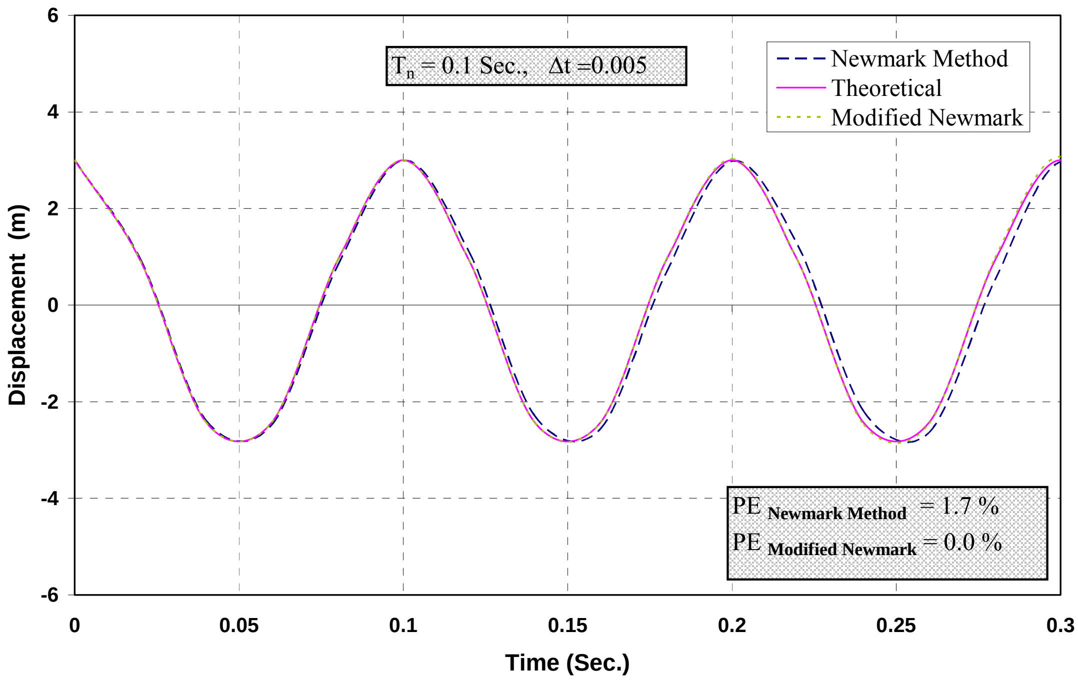


(e)



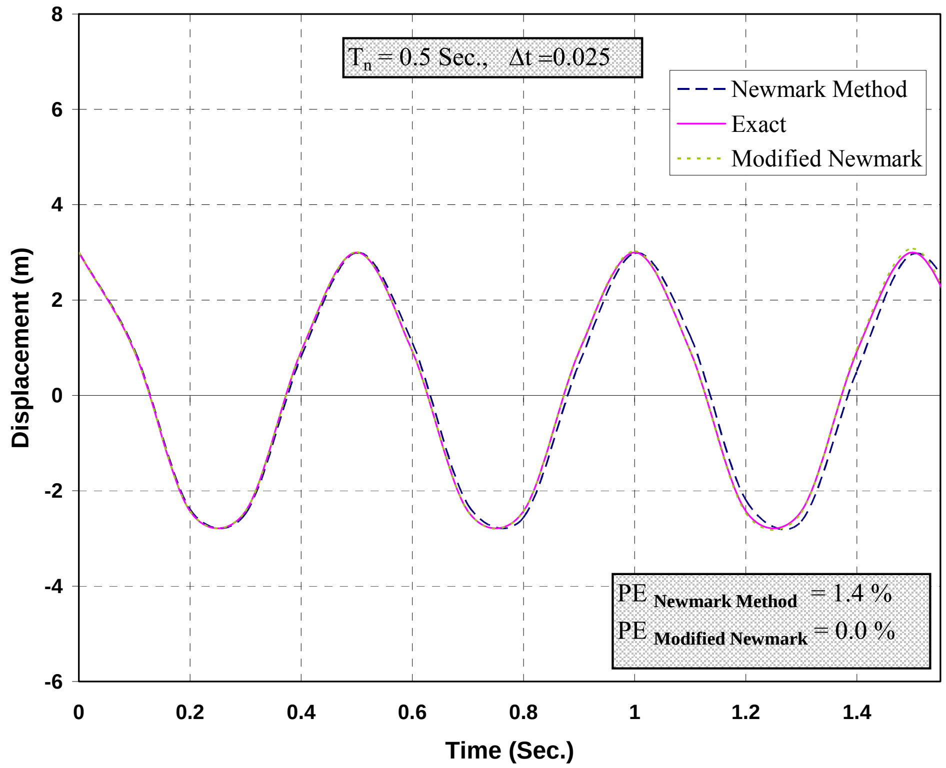
(f)

Figure (2): Response of damped SDF system to harmonic load using different periods T_n, damping ratios ζ and time steppings ΔT



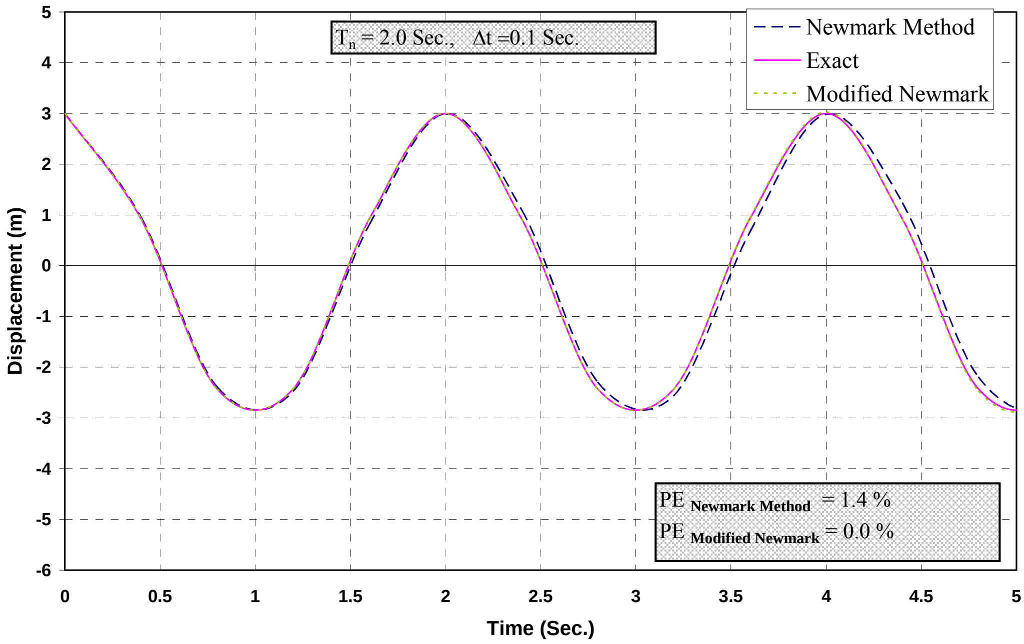
T_n = 0.1 Sec., Δt =0.005
Newmark Method
Theoretical
Modified Newmark
PE Newmark Method = 1.7 %
PE Modified Newmark = 0.0 %
Displacement (m)
Time (Sec.)

(a)

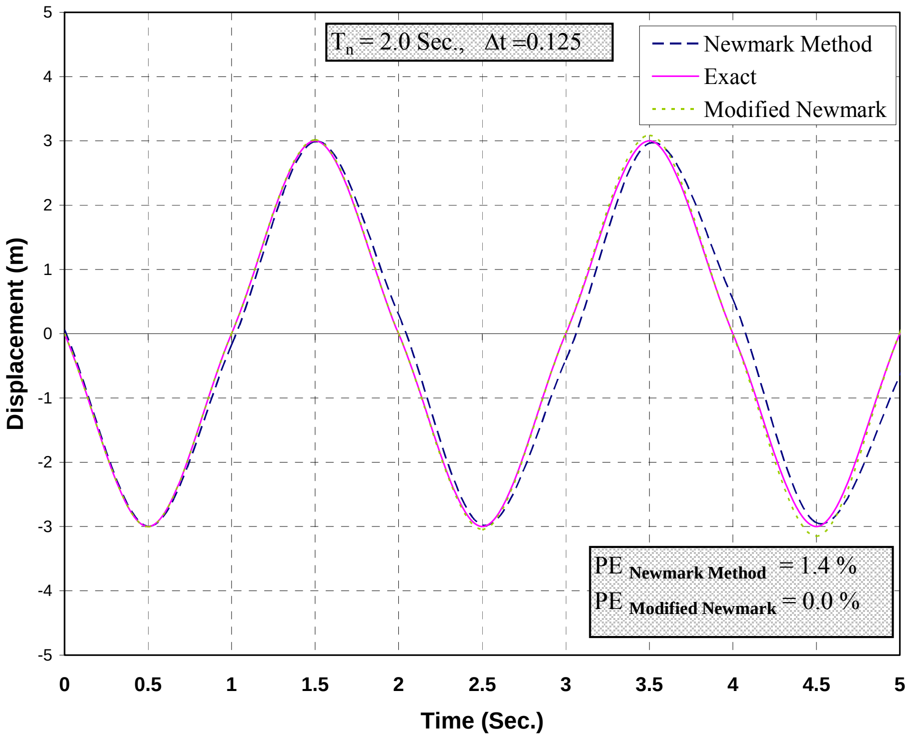


T_n = 0.5 Sec., Δt =0.025
Newmark Method
Exact
Modified Newmark
PE Newmark Method = 1.4 %
PE Modified Newmark = 0.0 %
Displacement (m)
Time (Sec.)

(b)



(c)



(d)

Figure (3): Predicted results of the proposed method compared with the exact solution and Newmark's method to estimate the percentage period error

REFERENCES

Bathe, K.J. (1996). "Finite element procedures". Upper Saddle River, Prentice Hall, NJ.

Bathe, K. J., and Wilson, E. L. (1976). "Numerical methods in finite element analysis". Prentice-Hall, Englewood Cliffs, NJ.

Bathe, K.J., and Noh, G. (2012) "Insight into an implicit time integration scheme for structural dynamics". Computers and Structures, 98-99, 1-6.

Chang, S.-Y. (2015). "Development and validation of a new family of computationally efficient methods for dynamic analysis." Journal of Earthquake Engineering, 19 (6), 847-873.

Chapra, S., and Canale, R. P. (1998). "Numerical methods for engineers". McGraw-Hill, New York, N.Y.

Chen, C., and Ricles, J. M. (2008). "Development of direct integration algorithms for structural dynamics using discrete control theory." Journal of Engineering Mechanics, 134 (8), 676-683.

Chopra, A.K. (2013). "Dynamics of structures". Prentice Hall, 4th Edition, Berkeley, University of California, USA.

Houbolt, J.C. (1950). "A recurrence matrix solution for the dynamic response of elastic aircraft." J. Aeronaut. Sci., 17, 540-550.

Kim, S., Cho, J.Y., and Kim, W.D. (1997). "From the trapezoidal rule to higher-order accurate and unconditionally stable time-integration method for structural dynamics." Comp. Methods Appl. Engng., 149, 73-88. DOI: 10.1016/S0045-7825(97)00061-3.

Kolay, C., and Ricles, J. M. (2014). "Development of a family of unconditionally stable explicit direct integration algorithms with controllable numerical energy dissipation." Earthquake Engineering and Structural Dynamics, 43 (9), 1361-1380.

Kolay, C., and Ricles, J. M. (2016). "Assessment of explicit and semi-explicit classes of model-based algorithms for direct integration in structural dynamics." International Journal for Numerical Methods in Engineering, 107(1), 49-73.

Kolay, C., and Ricles, J.M. (2017). "Improved explicit integration algorithms for structural dynamic analysis with unconditional stability and controllable numerical dissipation". Journal of Earthquake Engineering, 1-22.

Krieg, R.D. (1973). "Unconditional stability in numerical time integration methods." J. Appl. Mech., 40, 417-420. DOI:10.1115/1.3422999.

Kutta, W. (1901). "Beitrag zur Integration totaler Differential-gleichungen." Z. Math. Phys., 46, 435-531.

Laier, J.E. (2000). "Hermitian one-step numerical integration algorithm for structural dynamic equations." Adv. Engng. Software, 31, 203-206. DOI: 10.1016/S0965-9978(99)00040-X.

Namadchi, A.H., and Alamatian, J. (2016). "Explicit dynamic analysis using dynamic relaxation method". Computers and Structures, 175, 91-99.

Newmark, N. M. (1959). "A method of computation for structural dynamics." ASCE J. Engng. Mech. Div., 85, 67-94.

Owren, B., and Simonsen, H.H. (1995). "Alternative integration methods for problems in structural dynamics." Comp. Methods Appl. Mech. Engng., 122, 1-10. DOI: 10.1016/0045-7825(94)00717-2.

Subbaraj, K., and Dokainish, A. (1989a). "A survey of direct time-integration methods in computational structural dynamics-I. Explicit methods." Comp and Struc., 6, 1371-1386.DOI: 10.1016/0045-7949(89)90314-3.

Subbaraj, K., and Dokainish, A. (1989b). "A survey of direct time-integration methods in computational structural dynamics-II. Implicit methods." Comp and Struc., 6, 1387-1401. DOI: 10.1016/0045-7949(89)90315-5.

Williamson, E.B., and Rungamornrat, J. (2002). "Numerical analysis of dynamic stability under random excitation." Engng. Struc., 24, 479-490. DOI:10.1016/S0141-0296 (01)00114-6.

Wilson, E. L., Farhoomand, I., and Bathe, K.J. (1973). "Nonlinear dynamic analysis of complex structures." Earthq. Engng. Struct. Dyn., 1, 241-252. DOI: 10.1002/eqe.4290010305.