Mechanical Buckling Analysis of Functionally Graded Plates Using a New Refined Theory

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ABSTRACT

In the present article, equilibrium and stability equations of rectangular functionally graded plates are determined using the new hyperbolic shear deformation theory. Unlike any other theory, the theory presented gives rise to only four governing equations. Number of unknown functions involved is only four, as against five in case of simple shear deformation theories of Mindlin and Reissner (first shear deformation theory). It is assumed that the material properties vary with the power product form of thickness coordinate variable z. The equilibrium and stability equations are employed to study the buckling behavior of functionally graded plates with all edges simply supported and subjected to in-plane loading conditions. The results obtained are compared with their counterparts in the literature.

KEYWORDS: New plate theory, Shear deformable plate theory, Functionally graded plates, Mechanical buckling.

INTRODUCTION

Recent studies on new performance materials have treated new composite materials known as Functionally Graded Materials (FGMs). These are high-performance heat-resistant materials able to withstand ultrahigh temperatures and extremely large gradients used in spacecrafts and nuclear plants. FGMs are microscopically inhomogeneous composite materials in which the mechanical properties vary smoothly and continuously from one surface to the other (Suresh and Mortensen, 1998). These novel materials were first introduced by a group of scientists in Sendai, Japan in 1984 (Koizumi, 1997) and then developed by other scientists (Koizumi and Niino, 1995; Kaysser and Ilschner, 1995). Typically, these materials are made from a mixture of ceramics and metal. It is apparent from the literature that most of the research on FGMs has been restricted to thermal stress analysis, fracture mechanics and optimization. Very little work has been done that considers stability analysis, buckling and vibrational behavior of structures made of functionally graded materials.

FG plate problems deal with two main concepts: the modeling of the plate (plate theories) and the procedure of solution. Most commonly used plate theories can be classified into three main categories: thin plate theory (Leissa, 1973; Yu, 2008; Bouazza and Adda-Bedia, 2013; Meisam et al., 2009) (e.g. Kirchhoff theory or

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CPT), moderately thick plate theory (Reissner, 1945; Reissner, 1944; Mindlin, 1951; Bouazza et al., 2009; Bouazza et al., 2010; Praveen and Reddy, 1998) (e.g. first-order shear deformation plate theory of Mindlin or FSDT) and thick plate theory (Reddy and Phan, 1985; Reddy, 1984; Hanna and Leissa, 1994; Zhou et al., 2002) (e.g. third-order shear deformation plate theory of Reddy or TSDT, higher-order shear deformation plate theory or HSDT and three-dimensional (3-D) elasticity theory).

Different higher-order shear deformation plate theories (HSDTs) were proposed, including the second-order shear deformation formulation of Whitney and Sun (1973) and the third-order shear deformation theory of Lo et al. (1977) with 11 unknowns; Kant (1982) with six unknowns; Bhimaraddi and Stevens (1984) with five unknowns; Reddy (1985, 1984) with five unknowns and Hanna and Leissa (1994) with four unknowns. Ambartsumian (1958) proposed a transverse shear stress function in order to explain plate deformation. A similar method was used later by Soldatos and Timarci (1993) for dynamic analysis of laminated shells. Later, some new functions were proposed by Reddy (1984), Touratier (1991), Karama et al. (2003), Soldatos (1992) and Aydogdu (2009).

An alternative variational theorem; namely the Reissner Mixed Variational Theorem (RMVT), was proposed by Reissner (1984; 1986) for the analysis of laminated composite plates, where the displacement and transverse stress components are regarded as the primary variables and the in-plane stress components can be calculated by the primary variables that are determined. The related developments, ideas and evaluations based on the RMVT with regard to the modeling of multi-layered plates and shells were precisely described by Carrera (2001). Based on the RMVT, Murakami (1986) and Toledano and Murakami (1987) proposed the mixed laminated plate theories with first-order and higher-order zigzag displacement models, respectively, to improve the in-plane responses of the laminated composite plates. Carrera (2003) presented a historical review of zigzag theories of multi-layered plates and shells. Carrera (2000, 2000) also presented the assessments of mixed and classical theories on global and local responses of multi-layered orthotropic plates and concluded that the RMVT-based theories are superior to the PVD-based ones after implementations and comparisons among more than 40 theories were undertaken.

Recently, a two-variable refined plate theory (RPT) was first developed for isotropic plates by Shimpi (2002) and was extended to orthotropic plates by Shimpi and Patel (2006, 2006) and Kim et al. (2009) and was extended to functionally graded plates by Mechab et al. (2010) and Thai and Choi (2012).

In the present article, equilibrium and stability equations for the rectangular functionally graded plates are obtained on the basis of new hyperbolic shear deformation theory. Resulting equations are employed to analyze the buckling behavior and obtain the critical buckling loads. In order to establish the fundamental system of equations for the buckling analysis, we assume that the non-homogeneous mechanical properties are given by the power product form of coordinate variable z.

Problem Formulation

Material Properties

Consider a rectangular plate of total thickness h as shown in Fig. 1. The plate FGM is made of ceramic and metal and the material properties of FGM vary continuously across the thickness according to the following equations, which are the same as the equations proposed by Praveen and Reddy (1998).

$$E(z) = E_m + E_{cm} V_f(z) \qquad E_{cm} = E_c - E_m$$
$$v(z) = v_0 \tag{1}$$

For simplicity, Poisson's ratio of the plate is assumed to be constant in this study, as the effect of Poisson's ratio on deformation is much less than that of Young's modulus (Delale and Erdogan, 1983).

Subscripts m and c refer to properties of metal and ceramics, respectively and $V_f(z)$ is the volume fraction of the constituents which can mostly be defined by power–law functions (Yu, 2008; Bouazza and Adda-Bedia, 2013; Meisam et al., 2009; Bouazza et al., 2009; Praveen and Reddy, 1998). For power-law FGM, the volume fraction function is expressed as:

$$V_f(z) = \left(z/h + 1/2\right)^k \tag{2}$$

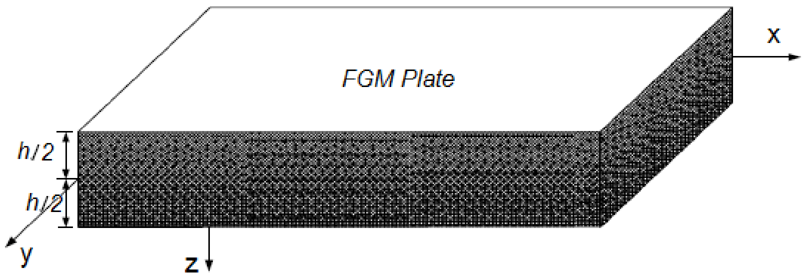

Figure (1): Typical FGM rectangular plate

Higher-Order Plate Theories with Five Unknown Functions

The displacements of a material point located at (x, y, z) in the plate may be written as:

$$\begin{aligned} U(x,y,z) &= u_0(x,y) - z\frac{\partial w_0}{\partial x} + \Psi(z)\theta_x \\ V(x,y,z) &= v_0(x,y) - z\frac{\partial w_0}{\partial y} + \Psi(z)\theta_y \\ W(x,y,z) &= w_0(x,y) \end{aligned} \tag{3}$$

where U, V and W are the displacements in the x, y and z directions, u_0, v_0 and w_0 are the midplane displacements and θ_x and θ_y are the rotations of the yz and xz planes due to bending, respectively. $\Psi(z)$ represents shape function determining the distribution of the transverse shear strains and stresses along the thickness. The description of various displacement models is given in Appendix1.

Present New Hyperbolic Shear Deformation Theory

Unlike the other theories, the number of unknown functions involved in the present refined hyperbolic shear deformation theory is only four, as against five in case of other shear deformation theories (Reissner, 1984; Reissner, 1986; Carrera, 2001; Murakami, 1986; Toledano and Murakami, 1987; Carrera, 2003; Carrera, 2000; Carrera, 2000). The theory presented is variationally consistent, does not require shear correction factor and gives rise to transverse shear stress variation such that the transverse shear stresses vary parabolically across the thickness, satisfying shear stress free surface conditions.

Assumptions of the Present Plate Theory

Assumptions of the present theory are as follows:

(i) The displacements are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.

(ii) The transverse displacement W includes two components of bending w_b and shear w_s. Both these components are functions of coordinates x and y.

$$W(x,y,z) = w_b(x,y) + w_s(x,y) \tag{4}$$

(iii) The transverse normal stress σ_z is negligible in comparison with in-plane normal stresses σ_x and σ_y.

(iv) The in-plane displacements U and V consist of extension, bending and shear components.

$$U = u + u_b + u_s \quad \text{and} \quad V = v + v_b + v_s \tag{5}$$

The bending components u_b and v_b are assumed to be similar, respectively, to the displacements given by the classical plate theory. Therefore, the expression for u_b and v_b can be given as:

$$u_b = -z\frac{\partial w_b}{\partial x} \quad \text{and} \quad v_b = -z\frac{\partial w_b}{\partial y} \tag{6}$$

The shear components u_s and v_s give rise, in conjunction with w_s, to the parabolic variations of shear strains γ_{xz}, γ_{yz} and hence to shear stresses σ_{xz}, σ_{yz} through the thickness of the plate in such a way that shear stresses σ_{xz}, σ_{yz} are zero at the top and bottom faces of the plate. Consequently, the expression for u_s and v_s can be given as:

$$u_s = -f(z)\frac{\partial w_s}{\partial x} \text{ and } v_s = -f(z)\frac{\partial w_s}{\partial y} \quad (7)$$

Kinematics and Constitutive Equations

Based on the assumptions made in the preceding section, the displacement field of the present theory can be obtained using Eqs. (4)–(7) as:

$$\begin{aligned} U(x,y,z) &= u_0(x,y) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \\ V(x,y,z) &= v_0(x,y) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \\ W(x,y,z) &= w_b(x,y) + w_s(x,y) \end{aligned} \quad (8)$$

where:

$$f(z) = \frac{(h/\pi)\sinh\left(\frac{\pi}{h}z\right) - z}{[\cosh(\pi/2) - 1]} \quad (9)$$

It should be noted that unlike the first-order shear deformation theory, this theory does not require shear correction factors. The kinematic relations can be obtained as follows:

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + zk_x^b + f(z)k_x^s \\ \varepsilon_y &= \varepsilon_y^0 + zk_y^b + f(z)k_y^s \\ \gamma_{xy} &= \gamma_{xy}^0 + zk_{xy}^b + f(z)k_{xy}^s \\ \gamma_{yz} &= g(z)\gamma_{yz}^s \\ \gamma_{xz} &= g(z)\gamma_{xz}^s \\ \varepsilon_z &= 0 \end{aligned} \quad (10)$$

where:

$$\begin{aligned} &\varepsilon_x^0 = \frac{\partial u}{\partial x},\ k_x^b = -\frac{\partial^2 w_b}{\partial x^2},\ k_x^s = -\frac{\partial^2 w_s}{\partial x^2} \\ &\varepsilon_y^0 = \frac{\partial v}{\partial y},\ k_y^b = -\frac{\partial^2 w_b}{\partial y^2},\ k_y^s = -\frac{\partial^2 w_s}{\partial y^2} \\ &\gamma_{xy}^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ k_{xy}^b = -2\frac{\partial^2 w_b}{\partial x \partial y},\ k_{xy}^s = -2\frac{\partial^2 w_s}{\partial x \partial y} \\ &\gamma_{yz}^s = \frac{\partial w_s}{\partial y},\ \gamma_{xz}^s = \frac{\partial w_s}{\partial x} \\ &g(z) = 1 - f'(z),\ f'(z) = \frac{df(z)}{dz} \end{aligned} \quad (11)$$

The results of the mathematical derivations are given in Appendix 2.

The linear constitutive relations of an FGM plate can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad (12)$$

where:

$$Q_{11} = Q_{22} = \frac{E(z)}{1-\nu^2},\ Q_{12} = \nu Q_{11},\quad Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\nu)} \quad (13)$$

Governing Equations

The strain energy of the plate can be written as:

$$U = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\,dV = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{xz}\gamma_{xz}\right)dV \quad (14)$$

The principle of virtual work for the present problem may be expressed as follows:

$$\iint \Big[N_x\delta\varepsilon_x^0 + N_y\delta\varepsilon_y^0 + N_{xy}\delta\gamma_{xy}^0 + M_x^b\delta k_x^b + M_y^b\delta k_y^b + M_{xy}^b k_{xy}^b + M_x^s k_x^s + M_y^s k_y^s + M_{xy}^s k_{xy}^s + Q_{yz}\gamma_{yz} + Q_{xz}\gamma_{xz}\Big]dxdy = 0 \quad (15)$$

where (N_x, N_y, N_{xy}) denote the total in-plane force resultants, (M_x^b, M_y^b, M_{xy}^b), (M_x^s, M_y^s, M_{xy}^s) denote the total moment resultants and (Q_{xz}, Q_{yz}) are transverse shear stress resultants and they are defined as:

$$\begin{aligned}
(N_x, N_y, N_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})\, dz \\
(M_x^b, M_y^b, M_{xy}^b) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})\, zdz \\
(M_x^s, M_y^s, M_{xy}^s) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})\, f(z)dz \\
(Q_{xz}, Q_{yz}) &= \int_{-h/2}^{h/2} (\sigma_{xz}, \sigma_{yz})\, g(z)\, dz
\end{aligned} \tag{16}$$

Substituting Eq. (12) into Eq. (16) and integrating through the thickness of the plate, the stress resultants are given as:

$$\begin{Bmatrix} \{N\} \\ \{M^b\} \\ \{M^s\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [B^s] \\ [B] & [D] & [D^s] \\ [B^s] & [D^s] & [H^s] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{k^b\} \\ \{k^s\} \end{Bmatrix} \tag{17a}$$

$$\begin{Bmatrix} Q_{yz} \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} \tag{17b}$$

where:

$$\begin{aligned}
&\{N\} = \{N_x, N_y, N_{xy}\}^t, \quad \{M^b\} = \{M_x^b, M_y^b, M_{xy}^b\}^t, \quad \{M^s\} = \{M_x^s, M_y^s, M_{xy}^s\}^t \\
&\{\varepsilon^0\} = \{\varepsilon_x^0; \varepsilon_y^0; \gamma_{xy}^0\}^t, \quad \{k^b\} = \{k_x^b, k_y^b, k_{xy}^b\}^t, \quad \{k^s\} = \{k_x^s, k_y^s, k_{xy}^s\}^t
\end{aligned} \tag{18}$$

and $A_{ij}, B_{ij}, \ldots$ etc. are the plate stiffness, defined by:

$$\begin{aligned}
(A_{ij}, B_{ij}, D_{ij}, B_{ij}^s, D_{ij}^s, H_{ij}^s) &= \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2, f(z), zf(z), (f(z))^2)\, dz \quad (i,j = 1,2,6) \\
A_{ij}^s &= \int_{-h/2}^{h/2} Q_{ij}(g(z))^2\, dz \qquad (i,j = 4,5)
\end{aligned} \tag{19}$$

The stability equations of the plate may be derived by the adjacent equilibrium criterion. Assume that the equilibrium state of the FGM plate under thermal loads is defined in terms of the displacement components $(u_0^0, v_0^0, w_b^0, w_s^0)$. The displacement components of a neighboring stable state differ by $(u_0^1, v_0^1, w_b^1, w_s^1)$ with respect to the equilibrium position. Thus, the total displacements of a neighboring state are:

$$u_0 = u_0^0 + u_0^1, \; v_0 = v_0^0 + v_0^1, \; w_b = w_b^0 + w_b^1, \; w_s = w_s^0 + w_s^1 \tag{20}$$

where the superscript 1 refers to the state of stability and the superscript 0 refers to the state of equilibrium conditions.

Substituting Equations (10), (11) and (20) into Equation (15) and integrating by parts and then equating the coefficients of δu_0^1, δv_0^1, δw_b^1 and δw_s^1 to zero, separately, the governing stability equations are obtained for the shear deformation plate theories as:

$$\begin{aligned}
&\frac{\partial N_x^1}{\partial x} + \frac{\partial N_{xy}^1}{\partial y} = 0 \\
&\frac{\partial N_{xy}^1}{\partial x} + \frac{\partial N_y^1}{\partial y} = 0 \\
&\frac{\partial^2 M_x^{b1}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{b1}}{\partial x \partial y} + \frac{\partial^2 M_y^{b1}}{\partial y^2} + \overline{N} = 0 \\
&\frac{\partial^2 M_x^{s1}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{s1}}{\partial x \partial y} + \frac{\partial^2 M_y^{s1}}{\partial y^2} + \frac{\partial Q_{xz}^{s1}}{\partial x} + \frac{\partial Q_{yz}^{s1}}{\partial y} + \overline{N} = 0
\end{aligned} \tag{21}$$

with:

$$\overline{N} = N_x^0 \frac{\partial^2 (w_b^1 + w_s^1)}{\partial x^2} + N_y^0 \frac{\partial^2 (w_b^1 + w_s^1)}{\partial y^2} + 2N_{xy}^0 \frac{\partial^2 (w_b^1 + w_s^1)}{\partial x\, \partial y} \tag{22}$$

Clearly, when the effect of transverse shear deformation is neglected $(w_s = 0)$, the governing equation (21) yields the governing equation of FGM plate based on the classical plate theory.

Navier Solution for Simply Supported Rectangular Plates

Rectangular plates are generally classified in accordance with the type of support used. We are here concerned with the exact solution of Equations (21) for a simply supported FGM plate. The following boundary conditions are imposed for the present refined plate theory at the side edges:

$$v_0^1 = w_b^1 = w_s^1 = \frac{\partial w_s^1}{\partial y} = N_x^1 = M_x^{b1} = M_x^{s1} = 0, \quad at \ x = 0, a \tag{23a}$$

$$u_0^1 = w_b^1 = w_s^1 = \frac{\partial w_s^1}{\partial x} = N_y^1 = M_y^{b1} = M_y^{s1} = 0, \quad at \ y = 0, b \tag{23b}$$

Based on the Navier method, the following approximate solution is seen to satisfy both the differential equation and the boundary conditions:

$$\begin{Bmatrix} u_0^1 \\ v_0^1 \\ w_b^1 \\ w_s^1 \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}^1 \cos\lambda x \sin\mu y \\ V_{mn}^1 \sin\lambda x \sin\mu y \\ W_{bmn}^1 \sin\lambda x \sin\mu y \\ W_{smn}^1 \sin\lambda x \sin\mu y \end{Bmatrix} \tag{24}$$

where $U_{mn}^1, V_{mn}^1, W_{bmn}^1, W_{smn}^1$ are arbitrary parameters to be determined and $\lambda = m\pi/a$ and $\mu = n\pi/b$.

Substituting Eq. (24) into Eq.(21) and using the derivation of new Eq. (21) given in Appendix 3,

$$[S]\{\Delta\} = 0 \tag{25}$$

where $\{\Delta\}$ denotes the column

$$\{\Delta\} = \{U_{mn}^1, V_{mn}^1, W_{bmn}^1, W_{smn}^1\}^t \tag{26}$$

and $[S]$ is the symmetric matrix given by:

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{12} & s_{22} & s_{23} & s_{24} \\ s_{13} & s_{23} & s_{33} & s_{34} \\ s_{14} & s_{24} & s_{34} & s_{44} \end{bmatrix} \tag{27}$$

in which:

$$\begin{aligned}
&s_{11} = A_{11}\lambda^2 + A_{66}\mu^2, \quad s_{12} = \lambda\mu(A_{12} + A_{66}), \quad s_{22} = A_{66}\lambda^2 + A_{22}\mu^2, \\
&s_{13} = -\lambda\left[B_{11}\lambda^2 + (B_{12} + 2B_{66})\mu^2\right], \quad s_{14} = -\lambda\left[B_{11}^s\lambda^2 + (B_{12}^s + 2B_{66}^s)\mu^2\right], \\
&s_{23} = -\mu\left[(B_{12} + 2B_{66})\lambda^2 + B_{22}\mu^2\right], \quad s_{24} = -\mu\left[(B_{12}^s + 2B_{66}^s)\lambda^2 + B_{22}^s\mu^2\right], \\
&s_{33} = D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4 + N_x^0\lambda^2 + N_y^0\mu^2 \\
&s_{34} = D_{11}^s\lambda^4 + 2(D_{12}^s + 2D_{66}^s)\lambda^2\mu^2 + D_{22}^s\mu^4 + N_x^0\lambda^2 + N_y^0\mu^2 \\
&s_{44} = H_{11}^s\lambda^4 + 2(H_{12}^s + 2H_{66}^s)\lambda^2\mu^2 + H_{22}^s\mu^4 + A_{55}^s\lambda^2 + A_{44}^s\mu^2 + N_x^0\lambda^2 + N_y^0\mu^2
\end{aligned} \tag{28}$$

The results of the mathematical derivations are given in Appendix 3.

By applying the condensation approach to eliminate the in-plane displacements U_{mn}^1 and V_{mn}^1, Eq. (25) can be rewritten as:

$$[\bar{S}]\{\bar{\Delta}\} = 0 \tag{29}$$

where:

$$[\bar{S}] = \begin{bmatrix} \bar{S}_{33} & \bar{S}_{43} \\ \bar{S}_{43} & \bar{S}_{44} \end{bmatrix} \tag{30}$$

$$[\bar{\Delta}] = \begin{Bmatrix} W_{bmn} \\ W_{smn} \end{Bmatrix}$$

and

$$\bar{s}_{33} = s_{33} - s_{13}\frac{b_1}{b_0} - s_{23}\frac{b_2}{b_0} + N_x^0(\lambda^2+\mu^2), \bar{s}_{34} = s_{34} - s_{14}\frac{b_1}{b_0} - s_{24}\frac{b_2}{b_0} + N_x^0(\lambda^2+\mu^2),$$
$$\bar{s}_{43} = s_{34} - s_{13}\frac{b_3}{b_0} - s_{23}\frac{b_4}{b_0} + N_x^0(\lambda^2+\mu^2), \bar{s}_{44} = s_{44} - s_{14}\frac{b_3}{b_0} - s_{24}\frac{b_4}{b_0} + N_x^0(\lambda^2+\mu^2), \quad (31)$$
$$b_0 = s_{11}s_{22} - s_{12}^2, b_1 = s_{13}s_{22} - s_{12}s_{23}, b_2 = s_{11}s_{23} - s_{12}s_{13}, b_3 = s_{14}s_{22} - s_{12}s_{24}, b_4 = s_{11}s_{24} - s_{12}s_{14}$$

Mechanical Buckling Analysis

Consider a rectangular plate with the length a and width b which is subjected to in-plane loads. Therefore, the pre-buckling forces can be obtained using the equilibrium conditions as (Yu, 2008; Bouazza and Adda-Bedia, 2013; Meisam et al., 2009):

$$N_x^0 = \xi_1 N_0, \ N_y^0 = \xi_2 N_0, \ N_{xy}^0 = 0 \quad (32)$$

where N_0 is the force per unit length, ξ_1 and ξ_2 are the load parameters which indicate the loading conditions. Negative values for ξ_1 and ξ_2 indicate that the plate is subjected to biaxial compressive loads, while positive values are used for tensile loads. Also, zero value for ξ_1 or ξ_2 shows uniaxial loading in x or y directions, respectively, as shown in Fig. 2.

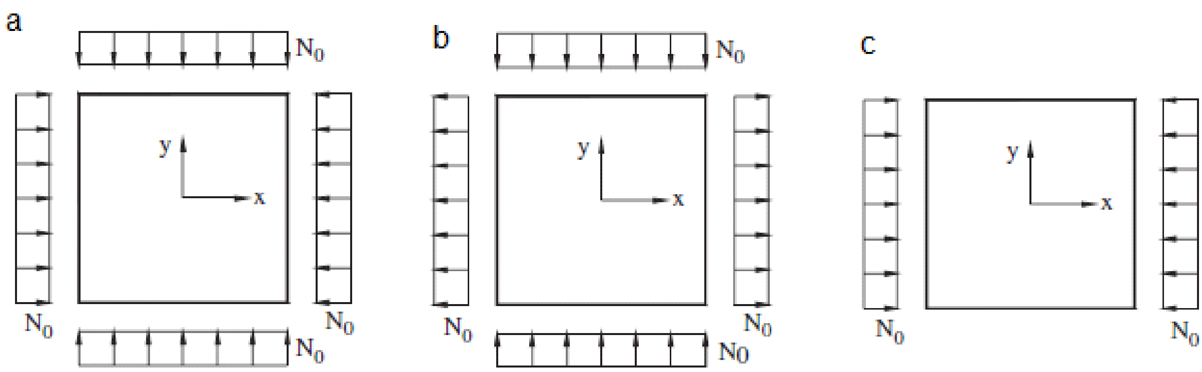

Figure (2): The loading conditions of rectangular plate for (a) biaxial compression (b) tension in the x direction and compression in the y direction and (c) uniaxial compression

Substituting Eq. (32) into Eq. (30), for non-trivial solution, the determinant of the coefficient matrix in new Eq. (30) must be zero. This gives the following expression for buckling load:

$$N_{cr}(m,n) = \frac{-1}{\xi_1\lambda^2 + \xi_2\mu^2}\frac{\bar{S}_{33}\bar{S}_{44} - \bar{S}_{34}\bar{S}_{43}}{\bar{S}_{33} + \bar{S}_{44} - \bar{S}_{34} - \bar{S}_{43}} \quad (33)$$

For each choice of m and n, there is a corresponsive unique value of N_{cr}. The critical buckling load is the smallest value of $N_{cr}(m, n)$.

RESULTS AND DISCUSSION

In this section, various numerical examples are presented and discussed for verifying the accuracy and efficiency of the present theory in predicting the critical buckling load of FGM plates subjected to in-plane loading. For the verification purpose, the results obtained by the present theory are compared with those found in the literature using CPT, FSDT and HSDT. The following material properties are used:

Material 1 (Al=Al_2O_3): $E_c = 380\ GPa, E_m = 70\ GPa, \nu = 0.3$

Material 2 (Al=SiC): $E_c = 420\ GPa, E_m = 70\ GPa, \nu = 0.3$

Comparisons

In order to validate the accuracy of the present method, a comparison has been carried out with previously published results by Mizusawa (1993), Hosseini-Hashemi et al. (2008), Shufrin and Eisenberger (2005) and Xiang (1993) for both thin (h/a=0.001) and moderately thick, simply supported square rectangular plates. Plates are subjected to monoaxial, in-plane compressive applied loads in the x $(\xi_1 = -1, \xi_2 = 0)$ and equal biaxial, in-plane compressive applied loads $(\xi_1 = -1, \xi_2 = -1)$. Comparison results are shown in Tables 1 and 2.

Table 1. Comparison study of buckling load parameters, $\tilde{N}_{cr}/\pi^2 = N_{cr}a^2/\pi^2 D$ for simply supported isotropic square plates subjected to monoaxial in-plane compressive applied loads in the x direction $(\xi_1 = -1, \xi_2 = 0)$

Method	h/a			
	0.001	**0.05**	**0.1**	**0.2**
Mizusawa (1993)	4	(3.928)	(3.729)	(3.119)
Mizusawa (1993)	4	3.944	3.784	3.256
Hosseini-Hashemi et al. (2008)[a]	4	3.9437	3.7838	3.2558
Shufrin and Eisenberger (2005)	–	–	3.7865	3.2637
Hosseini-Hashemi et al. (2008)[b]	4	3.9444	3.7864	3.2637
Present theory	4	3.9444	3.7868	3.2656

( ) Considering higher-order shear strain.
[a] Shear correction factor $k^2 = \pi^2/12$.
[b] Shear correction factor $k^2 = 5/6$.

Table 2. Comparison study of buckling load parameters, $\tilde{N}_{cr}/\pi^2 = N_{cr}a^2/\pi^2 D$, for simply supported isotropic square plates subjected to equal biaxial in-plane compressive applied loads $(\xi_1 = -1, \xi_2 = -1)$

Method	h/a				
	0.001	**0.005**	**0.05**	**0.1**	**0.2**
Xiang (1993)	2	–	1.9719	1.8920	1.7723
Hosseini-Hashemi et al. (2008)[a]	2	1.9997	1.9718	1.8919	1.7722
Present theory	2	1.9997	1.9722	1.8934	1.6328

[a] Shear correction factor $k^2 = 0.823045$.

The buckling load parameters $\tilde{N}_{cr}/\pi^2$ are listed in Table 1 for h/a= 0.001, 0.05, 0.1 and 0.2 together with the available results by Mizusawa (1993), obtained by using the spline strip method, by Shufrin and Eisenberger (2005), obtained by using the Levy method and Hosseini-Hashemi et al. (2008), obtained by exact solution of rectangular Mindlin plates. An excellent agreement can be observed between the present results and those given by Shufrin and Eisenberger (2005) and Hosseini-Hashemi et al. (2008). Good agreement has also been achieved between the present results and those of Mizusawa (1993). Note that higher-order terms (in the transverse shear strain) have been considered in Mizusawa's (1993) study. Those results are shown in parentheses in Table 1 to distinguish them from the rest of results. Full agreement of the present results for simply supported plates subjected to equal biaxial in-plane compressive loads $(\xi_1 = -1, \xi_2 = -1)$ exists with the solution of Xiang (1993), obtained by using the pb-2

Rayleigh–Ritz method and Hosseini-Hashemi et al. (2008) for shear correction factor $k^2 = 0.823045$. This is shown in Table 2 for both thin and moderately thick square plates. As Table 2 shows, the present results have good agreement with Hosseini-Hashemi et al. (2008) and Xiang (1993).

In addition, the comparison in Tables 1 and 2 shows that the results of the present theory have similar responses with both biaxial and uniaxial compression in all material cases. However, the critical buckling load parameters under biaxial compression are twice those under uniaxial compression.

The next comparison is performed for simply supported plates subjected to various loading conditions. The plate is made from a mixture of Aluminum (Al) and Silicon Carbide (SiC). The critical buckling loads of simply supported plate for different values of thickness ratio b/h, aspect ratio a/b and power law index k are shown in Table 3. It can be seen that the critical buckling load of the present theory is almost identical with those reported by Bodaghi and Saidi (2010) based on HSDT and Thai and Choi (2012) based on RPT. The changes of critical buckling mode of FGM plate predicted by the present theory as well as by HSDT and RPT are identical. In addition, it can be seen that the critical buckling load decreases as the power law index increases and the variation of the critical buckling load is considerable when the power law index is small. This is due to the fact that higher values of power law index correspond to high portion of metal in comparison with the ceramic part. In all material cases, the critical buckling increases when the geometric parameter b/h is increased. Moreover, increasing thickness ratio not only increases the values of critical buckling load, but also causes the changes in critical buckling mode. On the other hand, the critical buckling plate load under uniaxial compression $(\xi_1 = -1, \xi_2 = 0)$ is greater than that under biaxial compression $(\xi_1 = -1, \xi_2 = -1)$ and less than that under biaxial compression and tension $(\xi_1 = -1, \xi_2 = 1)$.

It should be noted that the present theory involves only four independent variables as against five in the case of HSDT (1993) and FSDT (Misuzawa, 1993; Hosseini-Hashemi et al., 2008; Shufrin and Eisenberger, 2005; Xiang, 1993). Also, the present theory does not require shear correction factors as in the case of FSDT. It can be concluded that the present theory is not only accurate, but also efficient in predicting the critical buckling loads of FG plates compared to other shear deformation plate theories, such as FSDT and HSDT.

Table 3. Comparison of critical buckling load (MN/m) of simply supported Al/SiC plate

(ξ_1, ξ_2)	a/b	b/h	Method	k = 0	k = 1	k = 2
(-1,0)	0.5	10	HSDT [49]	2079.721	1028.412	780.097
			RPT [43]	2079.758	1028.449	780.023
			Present	2079.926	1028.519	780.489
		5	HSDT [49]	12162.119	6270.298	4692.542
			RPT [43]	12164.987	6272.425	4695.029
			Present	12164.156	6272.046	4698.211
(-1,0)	1	10	HSDT [49]	1437.361	702.304	534.441
			RPT [43]	1437.389	702.251	534.835
			Present	1437.453	702.277	534.917
		5	HSDT [49]	9915.620	4955.431	3746.054
			RPT [43]	9916.193	4955.484	3746.732
			Present	9917.076	4955.862	3748.490
		10	HSDT [49]	1527.903[a]	748.920[a]	569.751[a]
			RPT [43]	1527.994[a]	748.988[a]	569.825[a]

(-1,0)	1.5		Present	1528.080[a]	749.023[a]	569.942[a]
		5	HSDT [49]	10044.721[a]	5067.219[a]	3819.109[a]
			RPT [43]	10044.962[a]	5068.084[a]	3820.079[a]
			Present	10045.701[a]	5068.407[a]	3822.248[a]
(-1,-1)	0.5	10	HSDT [49]	1663.777	822.738	624.158
			RPT [43]	1663.807	822.759	624.182
			Present	1663.941	822.815	624.391
		5	HSDT [49]	9729.999	5016.384	3754.274
			RPT [43]	9731.990	5017.941	3756.023
			Present	9731.325	5017.637	3758.569
(-1,-1)	1	10	HSDT [49]	718.692	351.124	267.416
			RPT [43]	718.695	351.125	267.418
			Present	718.726	351.138	267.459
		5	HSDT [49]	4957.888	2477.589	1873.190
			RPT [43]	4958.097	2477.742	1873.366
			Present	4958.538	2477.931	1874.245
(-1,-1)	1.5	10	HSDT [49]	526.861	256.776	195.714
			RPT [43]	526.862	256.776	195.714
			Present	526.879	256.784	195.737
		5	HSDT [49]	3772.877	1871.038	1418.120
			RPT [43]	3772.964	1871.101	1418.193
			Present	3773.286	1871.237	1418.723
(-1,1)	0.5	10	HSDT [49]	2772.980	1371.653	1040.519
			RPT [43]	2773.011	1371.265	1040.304
			Present	2773.235	1371.359	1040.652
		5	HSDT [49]	16216.712	8360.541	6257.811
			RPT [43]	16219.983	8363.233	6260.038
			Present	16218.875	8362.728	6264.282
(-1,1)	1	10	HSDT [49]	2772.980[a]	1371.653[a]	1040.519[a]
			RPT [43]	2773.011[a]	1371.265[a]	1040.304[a]
			Present	2773.235[a]	1371.359[a]	1040.652[a]
		5	HSDT [49]	16216.712[a]	8360.541[a]	6257.811[a]
			RPT [43]	16219.983[a]	8363.233[a]	6260.038[a]
			Present	16218.875[a]	8362.728[a]	6264.282[a]
(-1,1)	1.5	10	HSDT [49]	2772.980[b]	1371.653[b]	1040.519[b]
			RPT [43]	2773.011[b]	1371.265[b]	1040.304[b]
			Present	2773.235[b]	1371.359[b]	1040.652[b]
		5	HSDT [49]	16216.712[b]	8360.541[b]	6257.811[b]
			RPT [43]	16219.983[b]	8363.233[b]	6260.038[b]
			Present	16218.875[b]	8362.728[b]	6264.282[b]

a Mode for plate is (m, n) = (2, 1).

b Mode for plate is (m, n) = (3, 1).

CONCLUSIONS

Mechanical buckling analysis of simply supported FGM plates has been analyzed using a new four-variable refined plate theory. The number of primary variables in this theory is even less than that of first- and higher-order shear deformation plate theories. The theory gives a parabolic distribution of transverse shear strains and satisfies the zero traction boundary conditions on the surfaces of the plate without using shear correction factors. The buckling analysis of FGM plates under different types of mechanical loading is presented. Closed-form solutions for the critical buckling of plates are presented. It is concluded that:

1. It is a displacement-based theory that includes transverse shear effects.
2. The critical buckling loadings obtained using the present theory (with four unknown functions) and other shear deformation plate theories (with five unknown functions) are almost identical.
3. Transverse shear stress satisfies perfectly zero shear stress boundary conditions on top and bottom surfaces of the plate.
4. The theory obviates the need of shear correction factor.
5. The critical buckling load for the functionally graded plates is reduced when the power law index k increases.
6. The critical buckling load of plate under uniaxial compression is greater than that under biaxial compression and less than that under biaxial compression and tension.

Appendix 1

The displacement field of the classical thin plate theory (CPT) is obtained easily by setting $\Psi(z)=0$. The displacement of the first-order shear deformation plate theory (FSDPT) is obtained by setting $\Psi(z)=z$ (Reissner, 1945; Reissner, 1944; Mindlin, 1951). Also, the displacement of parabolic shear deformation plate theory (PSDPT) of Reddy (1984) is obtained by setting:

$$\Psi(z)=z\left(1-\frac{4z^2}{3h^2}\right) \tag{1}$$

The sinusoidal shear deformation plate theory (SSDPT) of Touratier (1991) is obtained by setting:

$$\Psi(z)=\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) \tag{2}$$

In addition, the exponential shear deformation plate theory (ESDPT) of Karama et al. (2003) is obtained by setting:

$$\Psi(z)=ze^{-2(z/h)^2} \tag{3}$$

Appendix 2

$$g(z)=1-\frac{\cosh\left(\frac{\pi}{h}z\right)-1}{\left[\cosh(\pi/2)-1\right]}$$

$$f'(z)=\frac{\cosh\left(\frac{\pi}{h}z\right)-1}{\left[\cosh(\pi/2)-1\right]}$$

Appendix 3

$$s_{11}=A_{11}\frac{\partial^2 u}{\partial x^2}+A_{66}\frac{\partial^2 u}{\partial y^2}=A_{11}\frac{\partial^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\cos\lambda x\sin\mu y}{\partial x^2}+A_{66}\frac{\partial^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\cos\lambda x\sin\mu y}{\partial y^2}$$

$$s_{12} = (A_{12} + A_{66})\frac{\partial^2 v}{\partial x \partial y} = (A_{12} + A_{66})\frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin \lambda x \cos \mu y}{\partial x \partial y}$$

$$s_{13} = -B_{11}\frac{\partial^3 w_b}{\partial x^3} - (B_{12} + 2B_{66})\frac{\partial^3 w_b}{\partial x \partial y^2}$$

$$= -B_{11}\frac{\partial^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{b\,mn} \sin \lambda x \sin \mu y}{\partial x^3} - (B_{12} + 2B_{66})\frac{\partial^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{b\,mn} \sin \lambda x \sin \mu y}{\partial x \partial y^2}$$

$$s_{14} = -B_{11}^s\frac{\partial^3 w_s}{\partial x^3} - (B_{12}^s + 2B_{66}^s)\frac{\partial^3 w_s}{\partial x \partial y^2}$$

$$= -B_{11}^s\frac{\partial^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{s\,mn} \sin \lambda x \sin \mu y}{\partial x^3} - (B_{12}^s + 2B_{66}^s)\frac{\partial^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{s\,mn} \sin \lambda x \sin \mu y}{\partial x \partial y^2}$$

$$s_{22} = A_{66}\frac{\partial^2 v}{\partial x^2} + A_{22}\frac{\partial^2 v}{\partial y^2} = A_{66}\frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin \lambda x \cos \mu y}{\partial x^2} + A_{22}\frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin \lambda x \cos \mu y}{\partial y^2}$$

$$s_{23} = -B_{22}\frac{\partial^3 w_b}{\partial y^3} - (B_{12} + 2B_{66})\frac{\partial^3 w_b}{\partial x^2 \partial y}$$

$$= -B_{22}\frac{\partial^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{b\,mn} \sin \lambda x \sin \mu y}{\partial y^3} - (B_{12} + 2B_{66})\frac{\partial^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{b\,mn} \sin \lambda x \sin \mu y}{\partial x^2 \partial y}$$

$$s_{24} = -B_{22}^s\frac{\partial^3 w_s}{\partial y^3} - (B_{12}^s + 2B_{66}^s)\frac{\partial^3 w_s}{\partial x^2 \partial y}$$

$$= -B_{22}^s\frac{\partial^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{s\,mn} \sin \lambda x \sin \mu y}{\partial y^3} - (B_{12}^s + 2B_{66}^s)\frac{\partial^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{s\,mn} \sin \lambda x \sin \mu y}{\partial x^2 \partial y}$$

$$s_{33} = -D_{11}\frac{\partial^4 w_b}{\partial x^4} - 2(D_{12} + 2D_{66})\frac{\partial^4 w_b}{\partial x^2 \partial y^2} - D_{22}\frac{\partial^4 w_b}{\partial y^4} + N_x^0 \frac{\partial^2 w_b}{\partial x^2} + N_y^0 \frac{\partial^2 w_b}{\partial y^2}$$

$$= -D_{11}\frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn} \sin \lambda x \sin \mu y}{\partial x^4} - 2(D_{12} + 2D_{66})\frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn} \sin \lambda x \sin \mu y}{\partial x^2 \partial y^2} -$$

$$D_{22}\frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn} \sin \lambda x \sin \mu y}{\partial y^4} + N_x^0 \frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn} \sin \lambda x \sin \mu y}{\partial x^2}$$

$$+ N_y^0 \frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn} \sin \lambda x \sin \mu y}{\partial y^2}$$

$$s_{34} = -D_{11}^s \frac{\partial^4 w_s}{\partial x^4} - 2(D_{12}^s + 2D_{66}^s)\frac{\partial^4 w_s}{\partial x^2 \partial y^2} - D_{22}^s \frac{\partial^4 w_s}{\partial y^4} + N_x^0 \frac{\partial^2 w_s}{\partial x^2} + N_y^0 \frac{\partial^2 w_s}{\partial y^2}$$

$$= -D_{11}^s \frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial x^4} - 2(D_{12}^s + 2D_{66}^s)\frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial x^2 \partial y^2}$$

$$- D_{22}^s \frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial y^4} + N_x^0 \frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial x^2}$$

$$+ N_y^0 \frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial y^2}$$

$$s_{44} = -H_{11}^s \frac{\partial^4 w_s}{\partial x^4} - 2(H_{12}^s + 2H_{66}^s)\frac{\partial^4 w_s}{\partial x^2 \partial y^2} - H_{22}^s \frac{\partial^4 w_s}{\partial y^4} + A_{55}^s \frac{\partial^2 w_s}{\partial x^2} + A_{44}^s \frac{\partial^2 w_s}{\partial y^2} + N_x^0 \frac{\partial^2 w_s}{\partial x^2} + N_y^0 \frac{\partial^2 w_s}{\partial y^2}$$

$$= -H_{11}^s \frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial x^4} - 2(H_{12}^s + 2H_{66}^s)\frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial x^2 \partial y^2}$$

$$- H_{22}^s \frac{\partial^4 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial y^4} + A_{55}^s \frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial x^2} + A_{44}^s \frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial y^2}$$

$$+ N_x^0 \frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial x^2} + N_y^0 \frac{\partial^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn} \sin \lambda x \sin \mu y}{\partial y^2}$$

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