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New Elastoplastic Method for Calculating the Contact Pressure Distribution under Rigid Foundations

***Waddah Salman Abdullah*[1)]**

[1)] Associate Prof., Department of Civil Engineering, Jordan University of Science and Technology, Irbid, P.O. Box 3030, Jordan, Waddah@just.edu.jo

ABSTRACT

An elastoplastic method for calculating the contact pressure distribution under rigid foundation has been developed and presented in this paper. The developed method is based on an elastoplastic approach. The first step is to find the point separating the elastic and plastic regions; i.e., the extent of the elastic and plastic yield zone under a specific load level. The contact pressure distribution is then found by combining a "modified" elastic distribution within the elastic zone and the contact pressure distribution within the plastic yield zone. An advanced elastoviscoplastic finite element analysis on rigid foundations was conducted to evaluate the accuracy of both the developed method and the Schultze method. It was found that the developed method yields better results than the Schultze method for all types of soils. The extent of plastic yielding using the developed method was found in complete agreement with the elastoviscoplastic finite element analysis. The Schultze method was found to overpredict the extent of plastic yielding underneath the foundation by about 12% at high load levels to 30% at low load levels.

KEYWORDS: Contact pressure, Rigid foundations, Elastic, Plastic, Viscoplasticity, Finite element method.

INTRODUCTION

Stress distribution induced by foundation loads at the soil-foundation interface represents the starting point for safe and economic design of foundations. Contact pressure distribution based on the elastic theory is not realistic in view of the fact that plastic yielding begins at the foundation's edges and spreads out toward the center of the foundation. Plastic yielding is initiated even at low load levels and is dependent on the type of the foundation's soil as well as the depth of foundation. Plastic yielding for foundations on surfaces of pure sandy soils is initiated immediately once load is applied on the foundation.

Normally practicing engineers assume constant contact pressure distribution under foundations. This assumption, besides its inaccuracy, is not on the safe side. The finite element method with nonlinear elastic or viscoplastic modeling may be used to determine very accurately the contact pressure distribution. However, the finite element method is a sophisticated and elaborate method which needs time, effort and experience. Therefore, the finite element method may not be available to all practicing engineers. Moreover, for small and medium sized tasks and due to its needed elaborate nature and required time and effort the finite element method may not be the right choice. Accordingly, there is a need for a method which accurately calculates the contact pressure distribution and requires less time and effort

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than the finite element method, and yet available for all practicing engineers.

Schultze in 1961 developed a method for calculating the contact pressure distribution. The method predicts the contact pressure distribution with reasonable accuracy. However, Schultze method overpredicts the amount of plastic yielding under the foundation and at all load levels by an average percentage of about 20%. The overprediction of Schultze method is the inevitable outcome of the assumption of linear distribution of pressure in the plastic zone. Overprediction of the plastic yield zone reduces the bending moment by reducing the arm of the resultant force of the contact pressure distribution rendering the RC foundation's design not on the safe side proportional to the amount of the overprediction.

This research work presents an elastoplastic method for calculating the contact pressure distribution under rigid foundations. Complete derivation of the new method is given in this paper. Examples for calculating the contact pressure distribution for various types of soils are given. Comparisons of the outcome of the developed method with Schultze method as well as results obtained from an advanced elastoviscoplastic finite element solution (Abdullah, 1982) were presented in this work for fair and just evaluation of the developed method and Schultze method.

SCOPE OF THIS WORK

The purposes of the present work are:

1. Developing a new elastoplastic method for predicting the contact pressure distribution.
2. Evaluating the developed method by using an advanced elastoviscoplastic finite element solution.
3. Evaluating the extent of plastic yielding as compared with finite element solution.

Elasto-Plastic Model for Contact Pressure Distribution

The two most important characteristics of an elastoplastic pressure distribution model are:

1. The extent of the region of plastic yielding underneath the foundation at a particular load level.
2. The model by which the contact pressure is distributed within the plastic region.

As a matter of fact, the model proposed for calculating the contact pressure distribution within the plastic zone directly influences and controls the extent of the plastic region underneath the foundation area. This fact is a direct consequence of the necessity of a statically admissible stress field underneath the foundation in equilibrium with the applied foundation load. The statically admissible stress field requires that the failure criterion should not be violated in any point within the soil domain under elastic equilibrium and represent a point on the plastic yield function such that in the plastic region:

$$F\ (\sigma_{i,j}) < 0 \quad \text{within the elastic region} \qquad (1)$$

$$F\ (\sigma_{i,j}) \geq 0 \quad \text{within the plastic region} \qquad (2)$$

$$P = \int_{-a}^{+a} \sigma_z\,(x)\,d\,x \qquad (3)$$

where F is the plastic yield function, $\sigma_{i,j}$ is the stress at a point in the soil domain, P is the applied load on the foundation, and $\sigma_z(x)$ is the vertical stress at the soil-foundation interface at point x away from the center of foundation, which also means the contact pressure.

The influence of the point of separation and the model for distributing the pressure distribution and *vice versa* would be discussed with necessary details when the proposed model would be introduced.

Foundation loads; due to the nature of (finite loaded area); are discontinuous outside the foundation area. The discontinuity of load causes high intensity of shear stresses at and also near the points of load discontinuity; i.e., at the edges of the foundation. High shear stress intensity at any point in soil is the source for plastic flow. In the plasticity and viscoplasticity terminology, high intensity of shear stresses at a point in soil causes violation of the plastic yield criterion. The level of violation of the plastic yield criterion is dependent on the extent by which the plastic yield surface was exceeded.

Therefore, the stresses induced in the supporting foundation soil and due to the applied loads are partly elastic and partly plastic. As the foundation load increases, the shear stresses increases accordingly and the zone of plastic yielding expands starting from the foundation edges towards the center of the foundation.

Finding the extent of the plastic zone represents a prerequisite for any elastoplastic model for calculating contact pressure distribution. The point of extension of the plastic region, also, represents the point of separation of the plastic region and the elastic region. The stress distribution within the elastic region is governed by the Boussinesq solution (Poulos and Davis, 1974). The stress distribution within the plastic region may, however, be determined by an appropriate model. Schultze proposed a linear distribution within the plastic region. In reality, the stress distribution within the plastic zone could never assume linear distribution. This fact is based on the results obtained from many observed measurements of pressure distributions made in many parts of the world by different investigators (Schultze, 1961). Moreover, assuming a linear distribution within the plastic region would necessitate prediction of larger extent of the plastic state region than the real situation.

THE PROPOSED ELASTOPLASTIC METHOD

The pressure distribution under rigid foundations is in fact not linear neither in the elastic region nor in the plastic region. Assuming it linear in either the elastic or plastic region introduces error in the contact pressure distribution. Initially, the location of the point of separation between the elastic region and plastic region would be affected. Assuming linear distribution in the plastic region causes overestimation of the extent of the plastic region. Consequently, the elastic region would be smaller than what it should be.

In order to obtain better estimation of the contact pressure distribution under rigid foundations, the point of separation of the elastic and plastic regions must be determined accurately. Moreover, the distribution of stresses within the plastic region must be improved by assuming a nonlinear model.

The characteristics of any elastoplastic contact pressure distribution are:

1. The extent to which the plastic region spreads out towards the center of the foundation; i.e., the point that separates the elastic and the plastic regions.
2. A model for expressing the elastic pressure distribution within the elastic region.
3. A model for expressing the plastic pressure distribution within the plastic region.
4. A mechanism for the redistribution of the excess stress in the plastic region to the neighboring elastic region.

Initially, the contact pressure distribution due to applied load on the foundation is elastic (Fig. 1). As the load level (which is the applied load divided by the ultimate load capacity) on the foundation increases, the shear stresses at the edges of the foundation increase accordingly. In soils; shear stresses are the source of plastic yielding. Therefore, when the shear stresses are built up due to the increase of load level, the plastic yield function would be violated $(F(\sigma_{i,j}) \geq 0)$. Plastic yielding in these areas develops and spreads out towards the center of the foundation as load level increases. Eventually, at ultimate load capacity *(*$q_{ult} \times B$*;* q_{ult} is the ultimate bearing capacity and B is the foundation width), where the load level is equal to 1, the stress distribution is totally plastic (Fig. 2).

The pressure distribution in the elastic region is governed by the well known Boussinesq distribution, as given by the following equation (Poulos and Davis, 1974):

$$\sigma_z(x) = \frac{P}{\pi a \sqrt{1 - \left(\frac{x}{a}\right)^2}} \tag{4}$$

Where $\sigma_z(x)$ is the contact pressure at point x away from the center of the foundation, P is the applied load on the foundation and "$a = B/2$" which is a half the foundation width (Fig. 1).

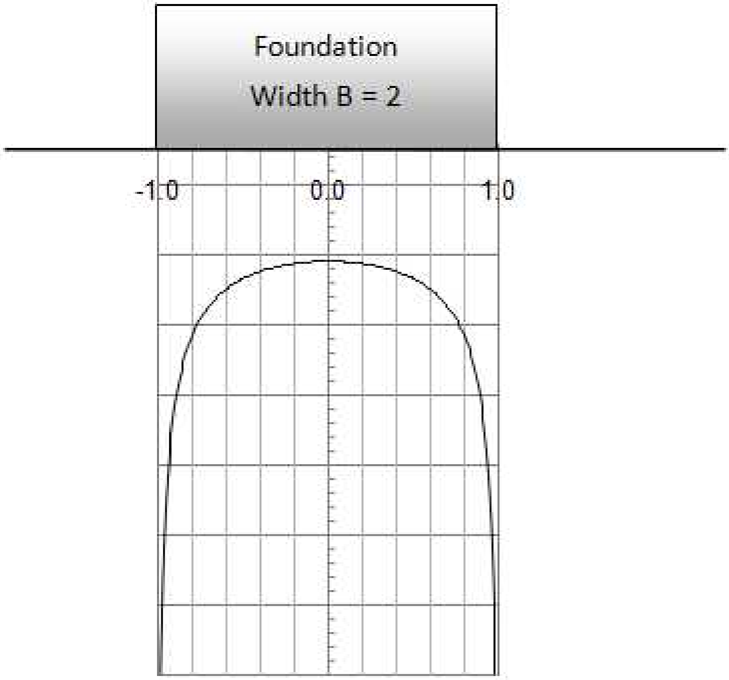

Fig. (1): Elastic contact pressure distribution under rigid foundation of width B = 2m.

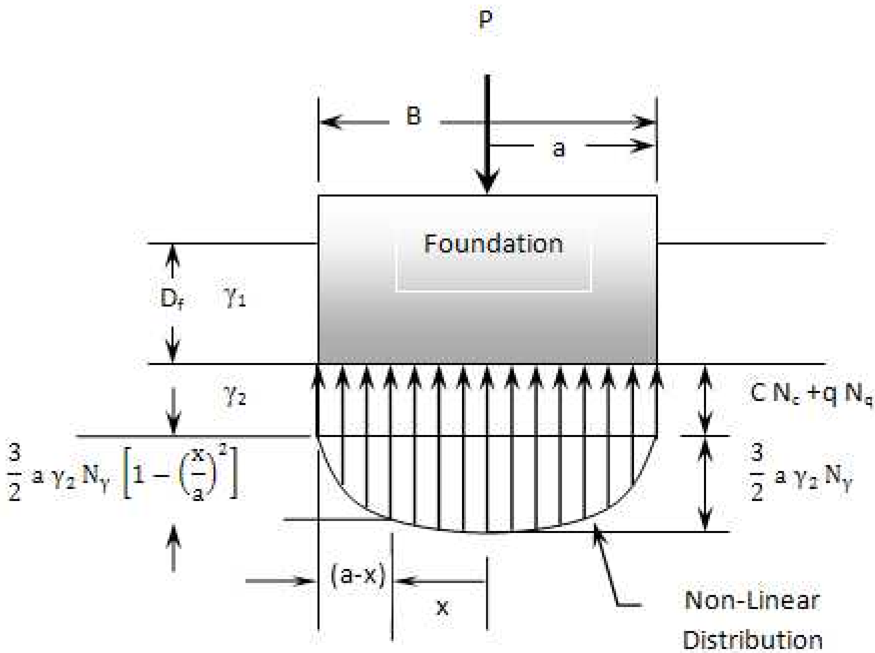

Fig. (2): Contact pressure distribution under rigid foundation at ultimate condition for a c – φ soil.

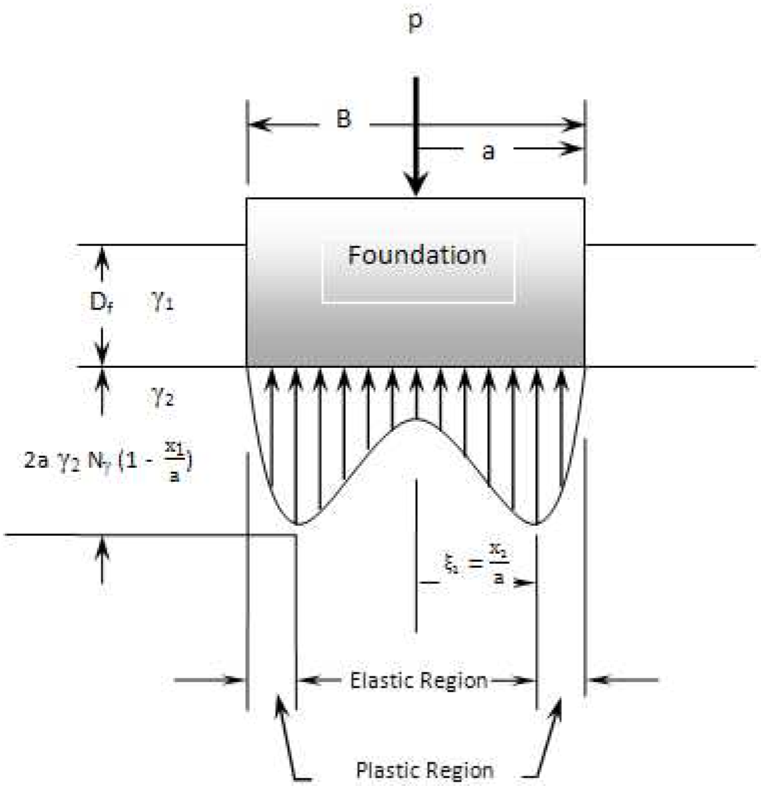

Fig. (3): Contact Pressure distribution under rigid foundation at load level below ultimate condition for a c – φ soil.

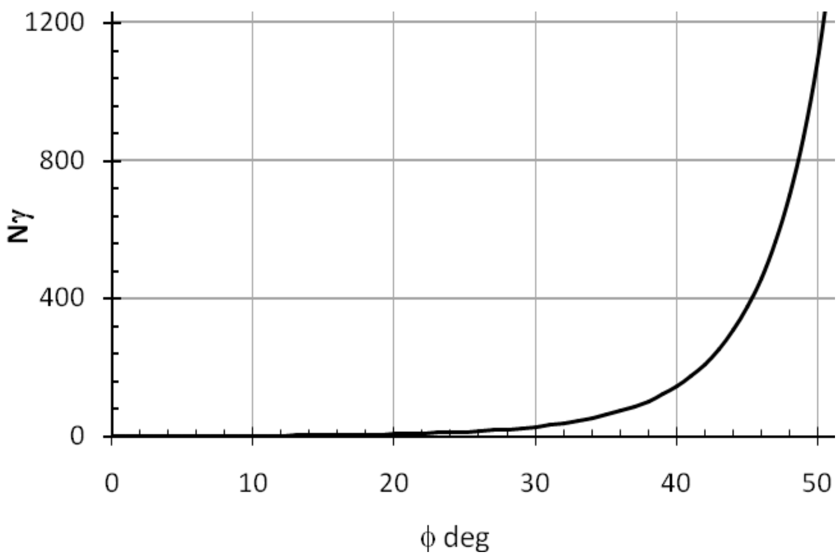

Fig. (4): Bearing capacity factor Nγ.

The Boussinesq distribution (Eq. 4) predicts an infinite value of pressure at the edges of the foundation $\left(\frac{x}{a}=1\right)$, and as can be demonstrated by Fig.1. Such high values of stresses at the edges can not be tolerated by the soil and thus, immediately, cause plastic yielding at these locations. The stresses at the edges can not exceed the ultimate strength, and hence should have a finite value of stress at these locations. The excess stress above the ultimate stress would then be redistributed to the neighboring regions. Obviously, redistributed excess stress would go to the elastic region and will be distributed according to an elastic formula. Following the redistribution process, the resulting elastoplastic contact pressure distribution looks as shown in Fig.3.

Development of the Model

As long as there is an exact solution in the elastic region (Boussinesq distribution represented by Eq.4), then it should be used to determine the contact pressure distribution in that region.

In the other extreme state; i.e., at the ultimate bearing capacity, there is no unique or exact solution to determine the plastic pressure distribution.

$$q_{ult} = c\,N_c + qN_q + \frac{1}{2}\gamma_2\,B\,N_\gamma \qquad (5)$$

where c is the soil cohesion, q is the overburden pressure at foundation level D_f, B is the width of foundation, γ_1, γ_2 are the unit weight above and below foundation level, respectively, N_c, N_q and N_γ are the bearing capacity parameters.

Schultze assumed linear distribution under ultimate condition; i.e., plastic equilibrium. As mentioned earlier, observations made on real foundations have demonstrated that the distribution in the plastic region is non-linear. The observed contact pressure distributions (Schultze, 1962) are, basically, quadratic. Therefore, the developed model adopted a quadratic model for the distribution in the plastic region.

Quadratic Distribution–Determination of Parameters

For a surface foundation on frictional soil (cohesion; c= 0, and depth of foundation; D_f =0), the ultimate load carrying capacity of the foundation is:

$$P = q_{ult} \times B \qquad (6)$$

where B is the foundation width and $B = 2\,a$, and q_{ult} is the ultimate bearing capacity of the foundation, given as:

$$q_{ult} = \frac{1}{2}\,\gamma_2\,B\,N_\gamma \qquad (7)$$

Hence, the ultimate load capacity of the foundation would be:

$$P = \left(\tfrac{1}{2}\gamma_2\,B\,N_\gamma\right) * B = \tfrac{1}{2}\gamma_2 \times 2\,a\,N_\gamma \times 2\,a = 2a^2\,\gamma_2\,N_\gamma \qquad (8)$$

The sought quadratic equation is of the form:

$$\sigma_z\,(x) = u\,x^2 + v \qquad (9)$$

where u and v are two quadratic constant parameters to be determined to fit the following two boundary conditions:

1. -he pressure at the edges are zero; i.e.,
2. The area covered by the quadratic distribution = ultimate load capacity of the foundation *"P"*.

$$P = 2\,a^2\,\gamma_2\,N_\gamma = \int_{x=-a}^{x=+a} 2\left(u\,x^2 + v\right)dx = 2\int_{x=-a}^{x=+a}\left(u\,x^2 + v\right)dx \qquad (10)$$

From the first boundary condition, we have:

$$u\;a^2 + v = 0 \qquad (11)$$

And from the second boundary condition, we have:

$$\int_0^{x=a}\left(u\,x^2 + v\right)dx = a^2\,\gamma_2\,N_\gamma \qquad (12)$$

$$\left[u\,\frac{x^3}{3} + v\,x\right]_{x=-a}^{x=+a} = a^2\,\gamma_2\,N_\gamma \qquad (13)$$

$$\left[u \frac{a^3}{3} + v \; a \right] = a^2 \, \gamma_2 \, N_\gamma \tag{14}$$

$$\left[u \frac{a^2}{3} + v \right] = a \, \gamma_2 \, N_\gamma \tag{15}$$

Solving Eq.(12) and Eq.(15) for the two quadratic parameters “u” and “v”, we get:

$$u = -\frac{3}{2a} \; \frac{\gamma_2 \, N_\gamma}{a} \tag{16}$$

$$v = \frac{3}{2} a \, \gamma_2 \; N_\gamma \tag{17}$$

$$\sigma_z \left(x\right) = \frac{3}{2} a \, \gamma_2 \, N_\gamma - \frac{3}{2a} \, \frac{\gamma_2 \, N_\gamma}{a} \; x^2 \tag{18}$$

or:

$$\sigma_z \left(x\right) = \frac{3}{2} a \, \gamma_2 \, N_\gamma \left[1 - \left(\frac{x}{a} \right)^2 \right] \tag{19}$$

By letting $\xi = \dfrac{x}{a}$, Equation 19 becomes:

$$\sigma_z \left(x\right) = \frac{3}{2} \; a \, \gamma_2 \, N_\gamma \left[1 - \xi^2 \right] \tag{20}$$

Finally, the contact pressure distribution at ultimate condition at any point below the foundation is given as:

$$\sigma_z \left(x\right) = c \, N_c + \, q \, N_q + \tfrac{3}{2} a \, \gamma_2 \, N_\gamma \left(1 - \xi^2\right) \tag{21}$$

Development of the Elastoplastic Model

In this part, the real state of stress will be considered; namely, the contact pressure distribution at a load level intermediate between the elastic state of stress where elastic state of stress under the foundation prevails (in equilibrium with the foundation load) and the state where plastic state of stress under the foundation is in equilibrium with the applied load on the foundation. In such case, it is anticipated that plastic yielding has developed the amount which is dependent on the load level. Let the extent of the elastic region be at a normalized distance $\xi_1 = x_1/a$ from the center of the foundation (Fig. 3).

Equilibrium of vertical stresses requires that load applied on the foundation equals the area of the contact pressure distribution. The area of the contact pressure distribution is the sum of the area under the elastic as well as the plastic pressure distributions. Since there is geometric symmetry and loading symmetry of the foundation, the derivation considers only half a of the contact pressure distribution.

The area under the pressure distribution in the elastic region is obtained from Eq.(4) and is given as:

$$A_{Elastic} = \frac{P_1}{\pi \, a} \int_0^{x_1} \frac{d \, x}{\sqrt{1 - \left(\frac{x}{a} \right)^2}} = \frac{P_1}{\pi} \sin^{-1} \xi_1 \tag{22}$$

The area under the pressure distribution in the plastic region is obtained from Eq.(21) and is given as:

$$A_{Plastic} = a \left(1 - \xi_1\right) \left(c \, N_c + q \, N_q\right) + \int_{x1}^{a} \; \tfrac{3}{2} \, a \, \gamma_2 N_\gamma \left(1 - \left(\tfrac{x}{a}\right)^2\right) dx \tag{23}$$

Integrating Eq. (23) yields:

$$A_{Plastic} = a \left(1 - \xi_1\right) \left(c \, N_c + q \, N_q\right) + \tfrac{1}{2} \, a^2 \, \gamma_2 \, N_\gamma \left(2 - 3 \, \xi_1 + \xi_1^{\,3}\right) \tag{24}$$

Thus the condition of equilibrium of contact pressure with the foundation load yields:

$$\frac{P}{2} = \frac{P_1}{\pi} \sin^{-1} \xi_1 + a\left(1 - \xi_1\right)\left(c \, N_c + q \, N_q\right) + \tfrac{1}{2} a^2 \, \gamma_2 \, N_\gamma \left(2 - 3\xi_1 + \xi_1^3\right) \tag{25}$$

where P_1 is the amount of foundation’s load needed to redistribute the excess stresses above yielding (stresses that caused yielding).

There are two unknowns to be determined, so that the full contact pressure distribution is totally known. These two values are the values of P_1 and ξ_1 separating the elastic and plastic regions. Continuity of the contact pressure distribution requires that the value of pressure at the interface of the elastic and the plastic regions should be the same; namely:

$$\sigma_z(\xi_1) = \frac{p_1}{\pi} \frac{1}{\sqrt{1 - \xi_1^2}} = \left[a \left(cN_c + qN_q\right) + \frac{3}{2} a^2 \, \gamma_2 \, N_\gamma (1 - \xi_1^2) \right] \tag{26}$$

$$P_1 = \pi \sqrt{1-\xi_1^2}\left[a\left(c\,N_c + q\,N_q\right) + \tfrac{3}{2}a^2\,\gamma_2\,N_\gamma\left(1-\xi_1^2\right)\right] \quad (27)$$

Substituting P_1 from Eq. 27 into Eq. 26 we get the following non-linear equation with only one unknown; namely ξ_1. The final equation becomes:

$$\frac{P}{2} = a\left(c\,N_c + qN_q\right)\left[\sqrt{1-\xi_1^2}\,\sin^{-1}\xi_1 + (1-\xi_1)\right] + a^2\gamma_2 N_\gamma\left[\frac{3}{2}(1-\xi_1^2)\sqrt{1-\xi_1^2}\,\sin^{-1}\xi_1 + \frac{1}{2}(2-3\xi_1+\xi_1^3)\right] \quad (28)$$

Eq.28 may be written in the standard form of nonlinear equations and thus we solve it for the unknown ξ_1 using the bisection method.

$$f(\xi_1) = a\left(c\,N_c + qN_q\right)\left[\sqrt{1-\xi_1^2}\,\sin^{-1}\xi_1 + (1-\xi_1)\right] + a^2\gamma\,N_\gamma\left[\frac{3}{2}(1-\xi_1^2)\sqrt{1-\xi_1^2}\,\sin^{-1}\xi_1 + \frac{1}{2}(2-3\xi_1+\xi_1^3)\right] - \frac{P}{2} \quad (29)$$

Once the value of ξ_1 is determined by solving Eq. 29, P_1 is determined using Eq. 27, the elastic pressure distribution is determined using Eq. 26 and the plastic distribution is determined using Eq. 26.

Comparison between the Developed Method and the Schultze Method

Two examples would be introduced for comparing the contact pressure distribution under rigid foundation obtained from the developed method and the Schultze method. The first example considers the contact pressure at various load levels for sandy soil. The other example considers the contact pressure at various load levels for mixed (c – φ) soil. The soil properties for both examples are given in Table 1. Obviously, the bearing capacity factors N_c, N_q and N_γ are required to calculate the bearing capacity of the foundation. For real soils (weight of soil is not ignored); there is no exact solution for the bearing capacity problem (Chen, 1975). Consequently, there are no unique values for the bearing capacity factors. The most debated factor as well as many investigators have developed methods for predicting it; is N_γ. For sandy soils, N_γ value varies significantly depending on the method of bearing capacity analysis and the assumed failure mechanism. For instance, Prandtl proposed three mechanisms with the resulting N_γ values differ significantly from one mechanism to the other. The used value for the N_γ was taken from Eq.(30) and Fig. 4, which was found to yield good results as was found from high accuracy finite element solutions. The bearing capacity factors used for the solved problems are given in Table (2).

$$N_\gamma = 2\left[1 + e^{\pi\tan(\phi)}\tan^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]\tan(\phi)\tan\left(\frac{\pi}{4}+\frac{\phi}{s}\right) \quad (30)$$

The extent of the plastic region as determined by the two methods was, also, a focus for comparison. As shown in Fig. 5, the extent of the plastic region is $1-\xi_1)$ thus the extents of the plastic region as predicted by the Schultze method and the developed method are $1-\xi_{1s})$ and $1-\xi_{1d})$, respectively. The Ratio of the extent of plastic region as determined by Schultze method to that determined by the developed method is calculated as:

$$\Re = \frac{\left(1-\xi_{1s}\right)}{\left(1-\xi_{1d}\right)} \quad (31)$$

For the pure φ-soil (first example) the contact pressure distributions obtained by the developed method and the Schultze method were compared at two different load levels (83% and 95%). The results are shown in Fig. 6 and Fig. 7, respectively. Load level is defined as the applied load divided by the ultimate load; i.e., the bearing capacity of the foundation. The Schultze method, as mentioned earlier, overpredicts the extent of the plastic region for all load levels. The extent of the plastic region predicted by both methods is given in Table (3).

The second example presents the contact pressure distribution of rigid strip foundation on c – φ soil. The width of the foundation and the soil properties are given in Table 1. The contact pressure distribution using both method at load levels, 75%, 83.3% and 95% are given in Fig. 8, Fig. 9 and Fig. 10, respectively. Yet again the Schultze method overpredicts the extent of the developed plastic region due to applied foundation's load as illustrated by Table (4).

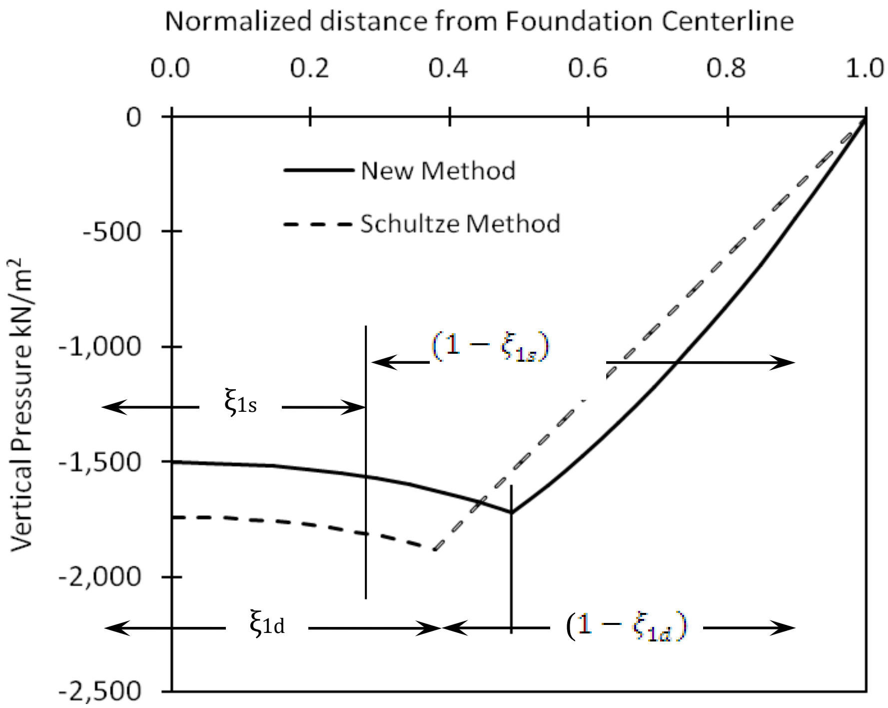

Fig. (5): Extent of the plastic region as determined by the developed method and Schultze method.

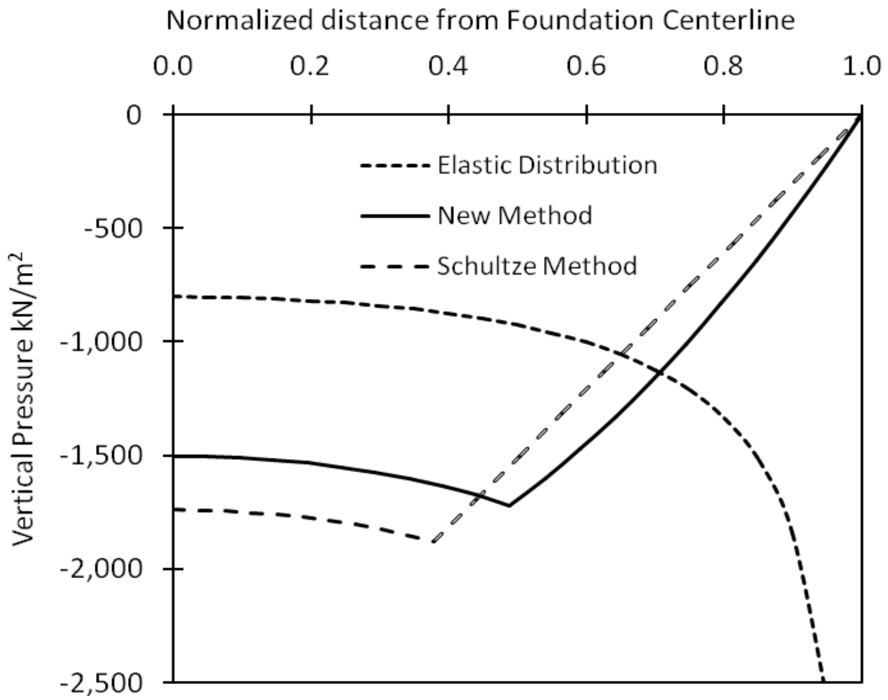

Fig. (6): Contact pressure distribution under rigid foundation on sandy soil, B = 1.2 m, $\phi = 40°$, $\gamma = 17.3$ kN/m^3, load level = 83%.

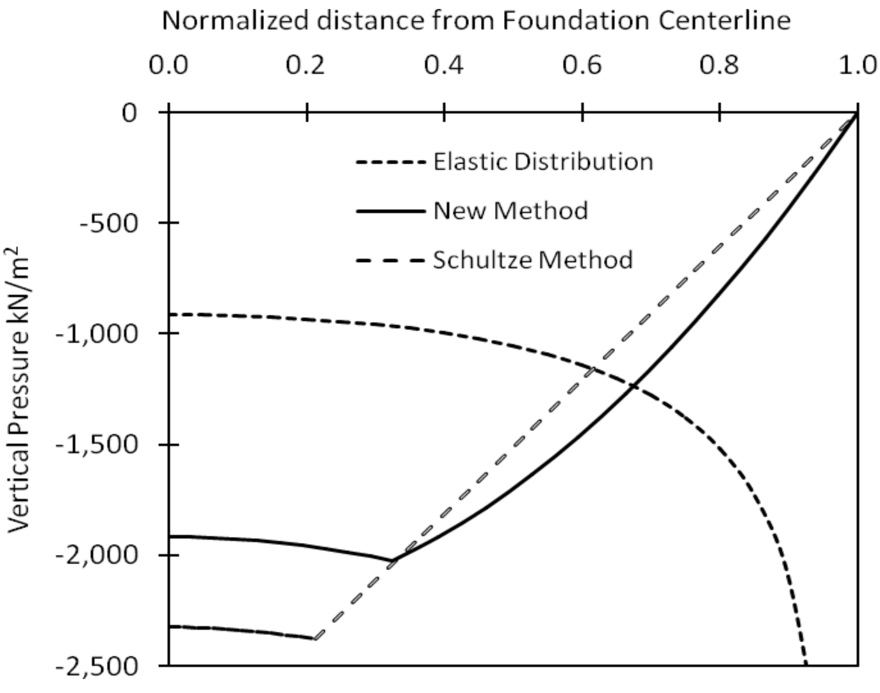

Fig. (7): Contact pressure distribution under rigid foundation on sandy soil, B = 1.2 m, $\phi = 40°$, $\gamma = 17.3$ kN/m^3, load level = 95%.

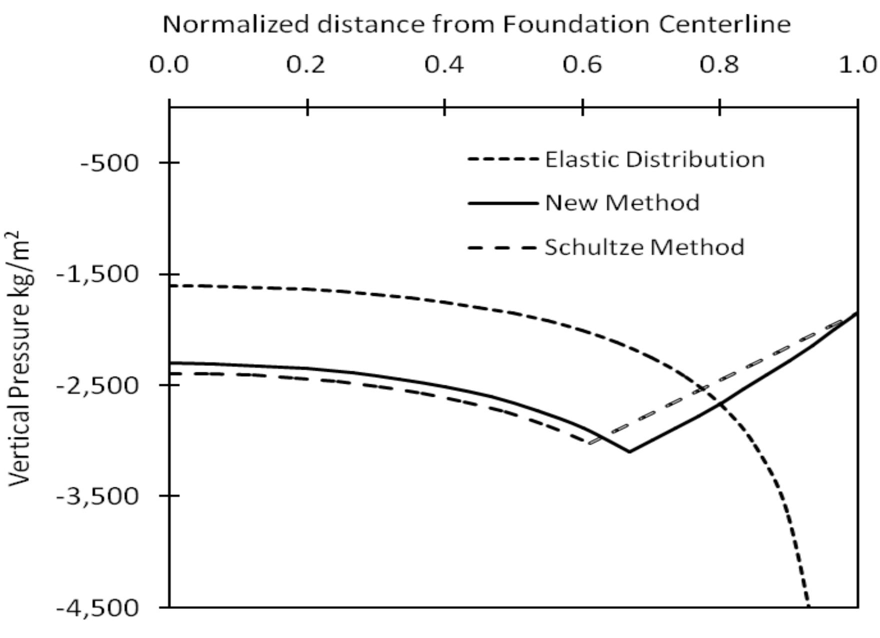

Fig. (8):Contact pressure distribution of rigid foundation on c- ϕ soil, c = 24.5 kN/m^2, $\phi = 40°$, γ= 17.3 kN/m^3, B = 1.2 m, load level = 75%.

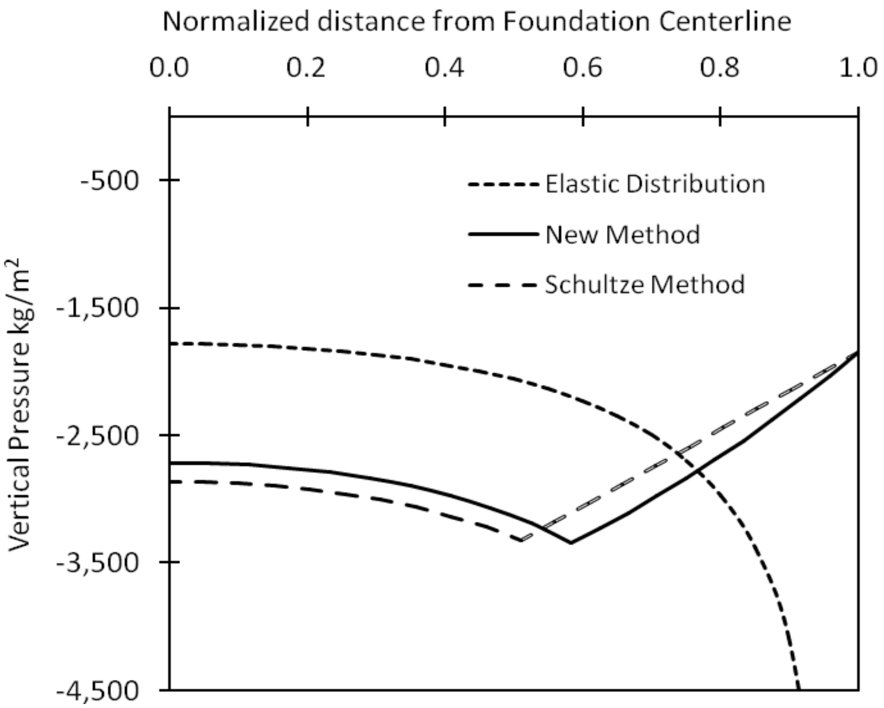

Fig. (9): Contact pressure distribution of rigid foundation on c- ϕ soil, c = 24.5 kN/m², ϕ = 40°, γ = 17.3 kN/m³, B = 1.2 m, load level = 83.3%.

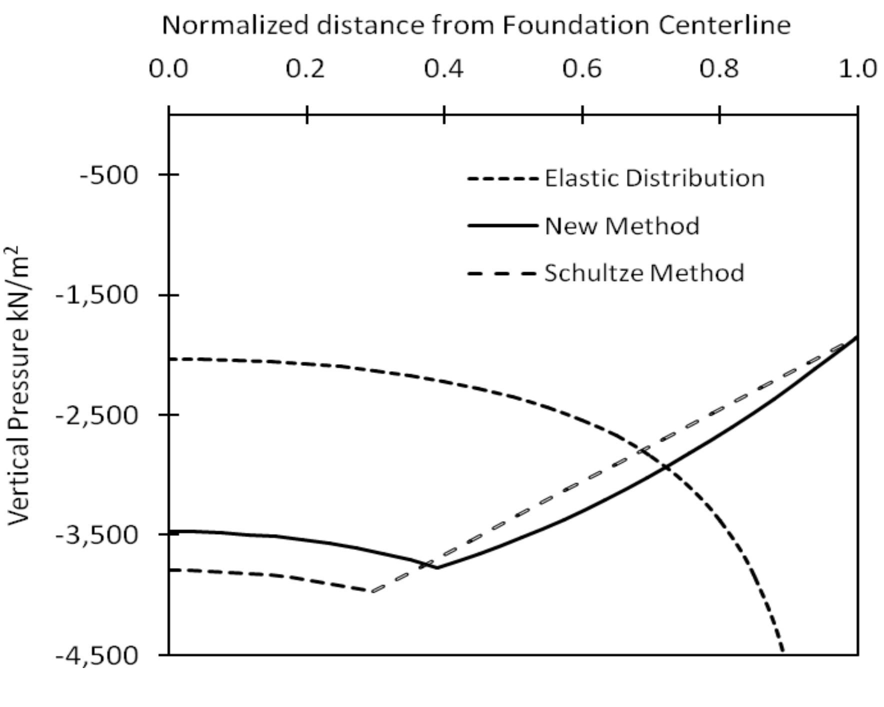

Fig. (10): Contact pressure distribution of rigid foundation on c - ϕ soil, c = 24.5 kN/m², ϕ = 40°, γ = 17.3 kN/m³, B = 1.2 m, load level = 95%.

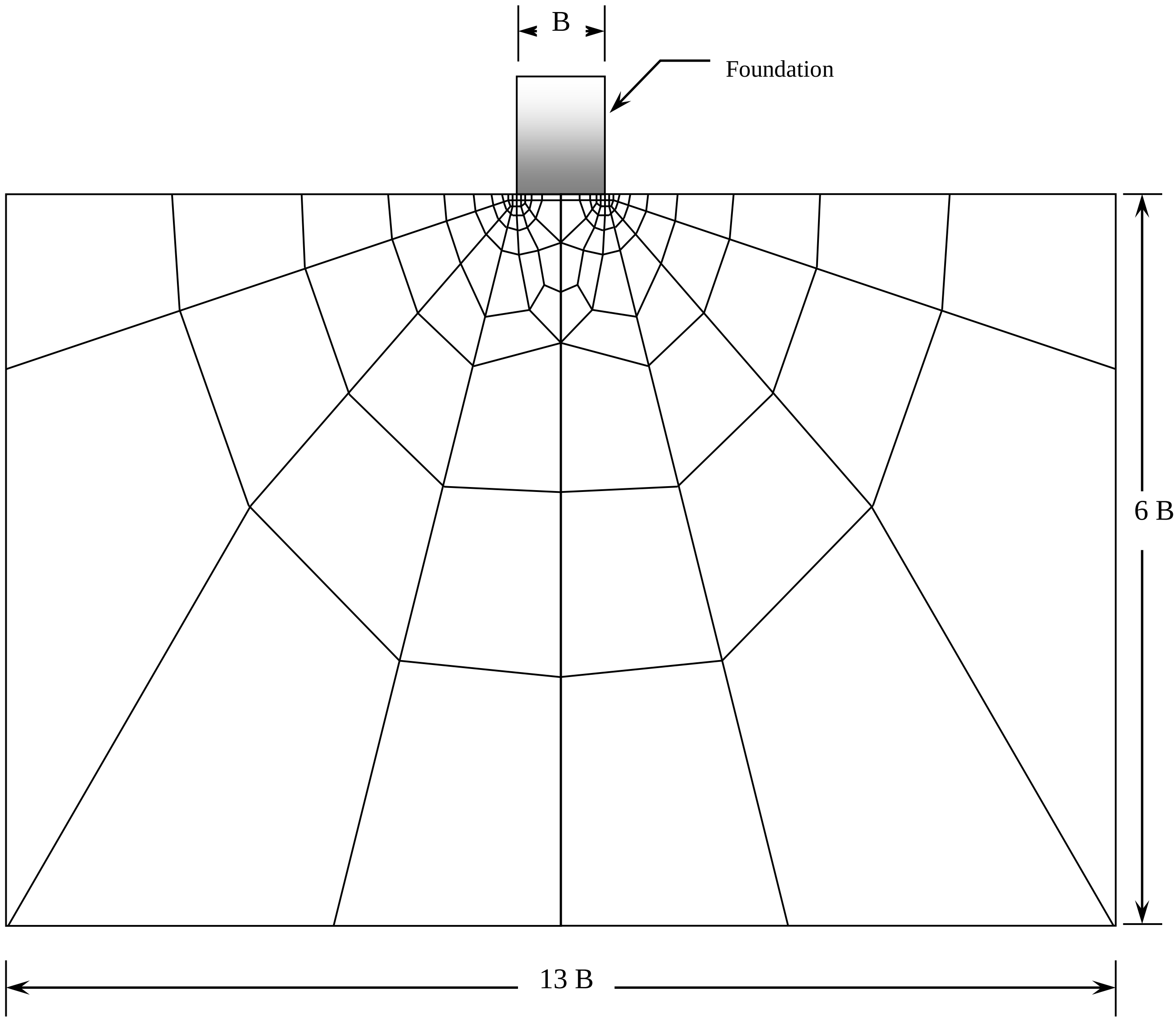

Fig. (11): Finite element idealization of the investigated domain.

Elasto-viscoplastic Finite Element Solution

The finite element method is a well developed method which can vigorously analyze complex and difficult engineering and scientific problems with high degree of exactness. The 8-noded isoparametric finite element was utilized for this analysis. The finite element domain for the analyzed foundation problem is shown in Fig. 11. The domain consists of 110; 8-noded isoparametric elements. Well known research workers (Zienkiewicz, 1977; Zienkiewicz and Taylor, 1989; Zienkiewicz et al., 2005) have found that the 8-noded isoparametric elements are the most appropriate elements for analysis as far as accuracy and stability of the results are concerned. Reduced integration was adopted using 2 by 2 Gauss integration points for each element.

Nonlinear finite element analysis was performed using the elastoviscoplasticity algorithm. Soil shear strength was modeled using Mohr-Coulomb failure criterion. Associated and non-associated flow rules were employed for calculating the viscoplastic strains.

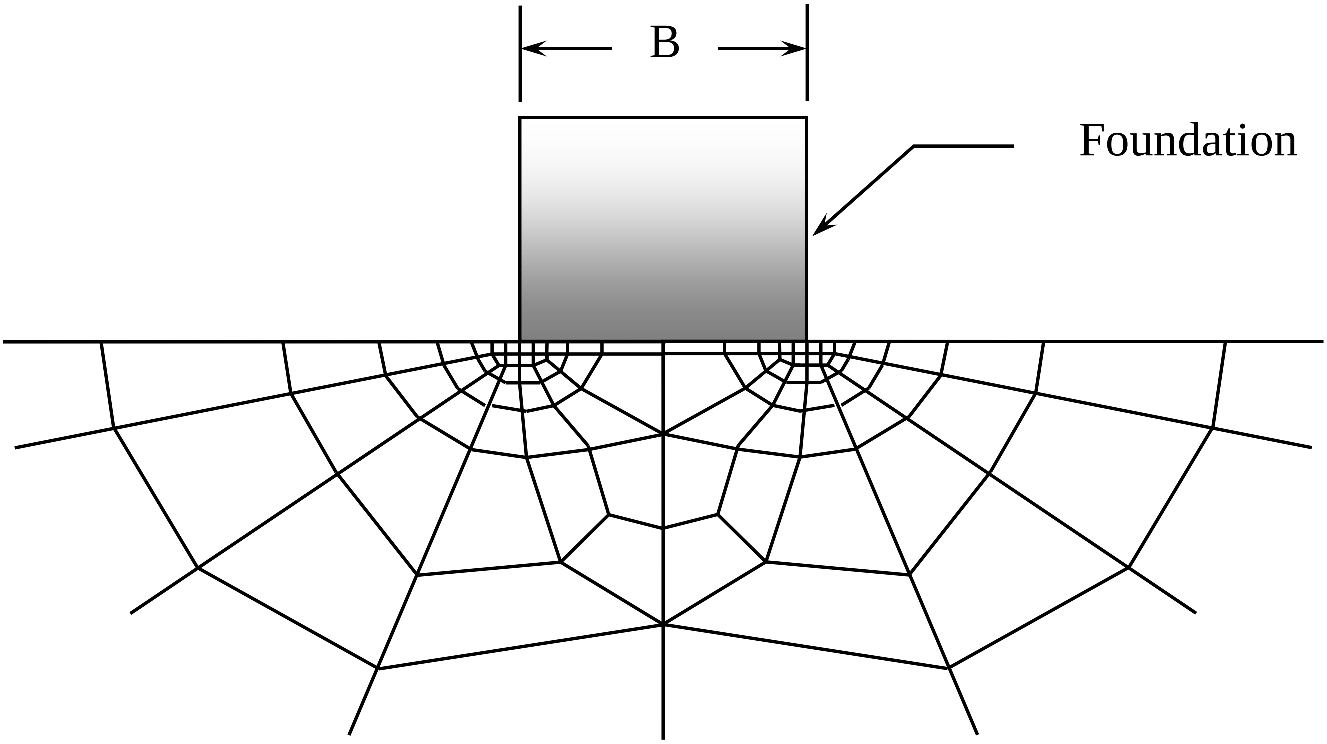

Fig. (12): Close-up of the finite element mesh in the vicinity of the foundation's edges.

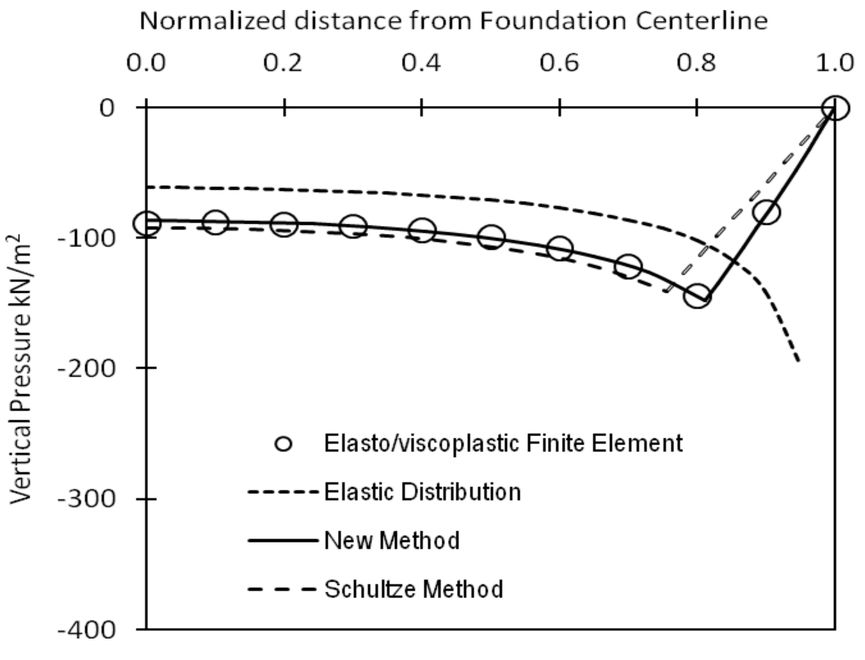

Fig. (13): Contact pressure distribution under rigid foundation on sandy soil, B = 1.524 m, $\phi = 40°$, $\gamma = 17.3$ kN/m^3, load level = 33.3%.

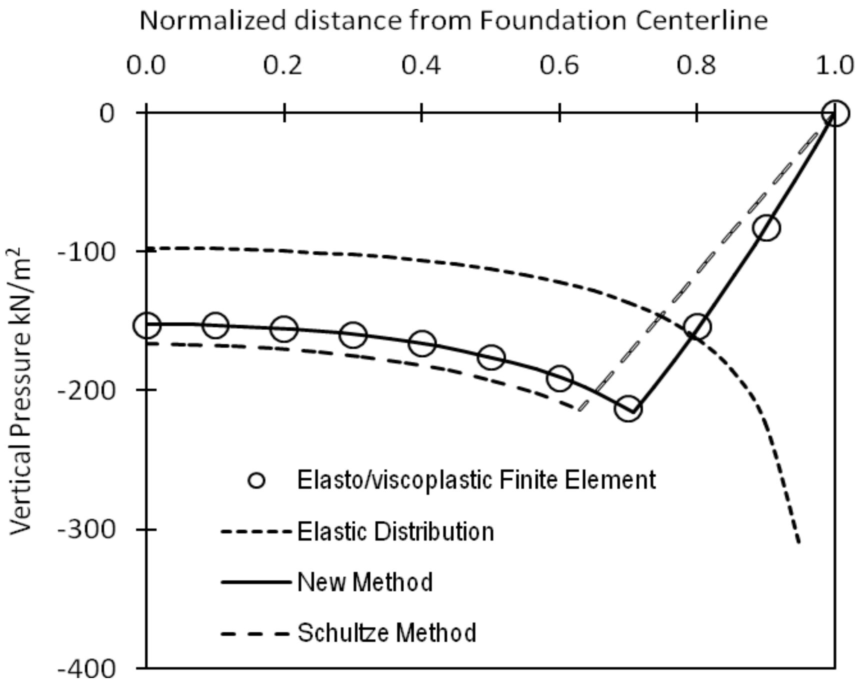

Fig. (14): Contact pressure distribution under rigid foundation on sandy soil, B = 1.524 m, $\phi = 40°$, $\gamma = 17.3$ kN/m^3, load level = 53%.

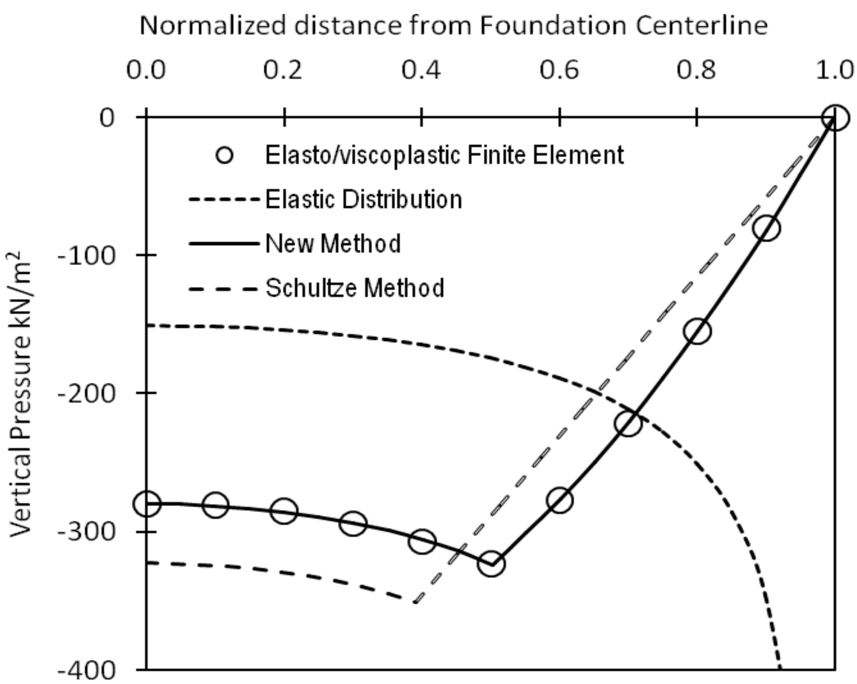

Fig. (15): Contact pressure distribution under rigid foundation on sandy soil, B = 1.524 m, $\phi = 40°$, $\gamma = 17.3$ kN/m^3, load level = 82%.

Conceptually, elastoviscoplasticity is more superior to the elastoplasticity in the sense that it can model soil behavior in real time, given the parameters of the soil are known. Otherwise, it may be used as a numerical artifice.

Details of the elastoviscoplasticity were given elsewhere (Abdullah, 1987a; Abdullah, 1987b; Abdullah, 1982). The basic idea of the elastoviscoplasticity is that soils (or any other materials) behave elastically as long as the failure criterion is not violated; $F(\sigma_{ij}) < 0$. Once plastic yielding is developed (i.e., failure criterion is violated $F(\sigma_{ij}) \geq 0$) in the soil domain, viscoplastic strain rate is developed $(\dot{\varepsilon}_{vp})$ whose value depends on the type of the plastic potential function and the amount of stresses by which the failure criterion was violated. The strain rate for an associated material is given as:

$$[\dot{\varepsilon}] = [D]^{-1}[\dot{\sigma}] + \gamma \left\langle \varphi(F)^{\delta} \right\rangle \frac{\partial F}{\partial[\sigma]} \tag{32}$$

and for a non-associated material (where $Q \neq F$) is given as:

$$[\dot{\varepsilon}] = [D]^{-1}[\dot{\sigma}] + \gamma \left\langle \varphi(F)^{\delta} \right\rangle \frac{\partial Q}{\partial[\sigma]} \tag{33}$$

where:
$[\dot{\varepsilon}]$ = total strain rates,
$[D]$ = elasticity matrix,
γ = fluidity parameter,
F = plastic yield function,
Q = plastic potential function,
$[\sigma]$ = stresses at a particular point in the soil domain, and
$[\dot{\sigma}]$= stress rates.
The square brackets $\langle \; \rangle$ mean:

$\left\langle \varphi(F)^{\delta} \right\rangle$ = 0.0 for F<0.0 failure criterion is not violated,

$\left\langle \varphi(F)^{\delta} \right\rangle = \varphi(F)^{\delta}$ for $F \geq 0.0$ failure criterion is violated.

When stresses at a particular point in the soil domain exceed the stresses allowed by the plastic yield criterion, the excess stresses would be redistributed viscoplastically to the surrounding region where stresses at any particular point are satisfied by $F(\sigma_{i,j}) < 0$. The viscoplastic strains are dissipated in time steps using Euler's marching scheme (Abdullah, 1987a; Abdullah, 1987b; Abdullah, 1982). Critical time stepping (Zienkiewicz and Cormeau, 1972, 1974) was used to ensure numerical stability of the solution. The time step for a Mohr-Coulomb plastic potential function is given by Zienkiewicz and Cormeau as:

$$\Delta t_{crit.} \leq \frac{(1+v)(1-2v)}{\gamma E(1-2v+\sin^2\varphi)} \tag{34}$$

where:
E = Modulus of elasticity, γ is the fluidity and υ is the Poison's ratio of the material.

For a Mohr-Coulomb material, the plastic yield function is given by the following function;

$$F = \sigma_m \sin\varphi + \bar{\sigma}\left\{\cos\theta_0 - \sin\theta_0 \frac{\sin\varphi}{\sqrt{3}}\right\} - c\cos\varphi = 0.0 \tag{35}$$

where:
φ = friction angle,
θ_0 = Lode angle, and is given as:

$$\theta_0 = \frac{1}{3}\sin^{-1}\left[-\frac{3\sqrt{3}}{2}\frac{J_s}{\bar{\sigma}^s}\right] \text{ with } -\frac{1}{6}\pi \leq \theta_0 \leq \frac{1}{6}\pi \tag{36}$$

$$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \tag{37}$$

$$\bar{\sigma} = \sqrt{\frac{1}{2}(S_x^2 + S_y^2 + S_z^2) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2} \tag{38}$$

$$S_x = \sigma_x - \sigma_m, \quad S_y = \sigma_y - \sigma_m, \quad S_z = \sigma_z - \sigma_m \text{ and}$$

$$J_3 = S_x S_y S_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - S_x\tau_{yz}^2 - S_y\tau_{xz}^2 - S_z\tau_{xy}^2 \tag{39}$$

Prandtl Solution Using Elastoviscoplastic Finite Element

Prior to calculating the contact pressure distribution using the elastoviscoplastic finite element approach, the accurateness of this approach was examined. The precision was examined by comparing the result obtained from the elastoviscoplastic finite element model with that

Table 1. Soil properties of the two considered examples.

Example Number	Foundation Width B (m)	Cohesion c (kN/m^2)	Friction Angle ϕ (°)	Unit Weight γ (kN/m^3)
1	1.2	0	40	17.3
2	1.2	24.5	40	17.3

Table 2. Bearing capacity factors for the solved problems.

Example Number	Friction Angle ϕ (°)	Cohesion c (kN/m^2)	Bearing Capacity Factors		
			N_c	N_q	N_γ
1	40	0	75.34	64.21	145.3
2	40	24.5	75.34	64.21	145.3
3	30	0	30.14	18.4	21.9

Table 3. Extent of the plastic region as determined by the developed method and the Schultze method for problem 1.

Load Level (%)	Schultze Method	Developed Method	$\Re = \frac{(1-\xi_{1s})}{(1-\xi_{1d})}$
83	0.378	0.489	1.22
95	0.216	0.325	1.17

Table 4. Extent of the plastic region as determined by the developed method and the Schultze method for problem 2.

Load Level (%)	Schultze Method	Developed Method	$\Re = \frac{(1-\xi_{1s})}{(1-\xi_{1d})}$
75	0.611	0.669	1.18
83.3	0.511	0.584	1.18
95	0.297	0.39	1.15

Table 5. Extent of the plastic region as determined by the developed method and the Schultze method for problem 3.

Load Level (%)	Schultze Method	Developed Method	$\Re = \frac{(1-\xi_{1s})}{(1-\xi_{1d})}$
33.3	0.756	0.811	1.3
53	0.629	0.709	1.28
82	0.392	0.503	1.22

of the Prandtl exact solution of a surface continuous foundation located on weightless cohesive soils. The predicted bearing capacity using the elastoviscoplastic finite element was only 1% above the Prandtl exact solution. This amount of error which was 0.01 of the exact solution may, even, be reduced further by specifying smaller tolerance value for the termination criterion of viscoplastic iteration scheme. Such high accuracy (compared to Prandtl exact solution) can not be achieved even by nonlinear elasticity finite element approach and can only be obtained from the elastoviscoplastic approach. This conclusion was reached from experience with other forms of nonlinear finite element analysis such as the nonlinear elasticity approach.

Adopting an accurate model for the analysis can not, in itself, assure accurate results unless the soil domain is accurately modeled and appropriately represent regions of sharp changes in stresses, particularly shear stresses. Therefore the finite element mesh represents the second most important (next to the elastoviscoplastic model) feature of a precise analysis. The mesh shown in Fig. 11 was carefully thought of and designed to achieve the required goals. This mesh idealizes the regions of sharp changes in stresses, at the edges of the foundation, with small finite elements. Fig. 12 furnishes a close up of the style of distribution of the finite element within critical regions of sharp changes in stresses.

Evaluating the Accuracy of the Developed Method and the Schultze Method

As demonstrated earlier, the developed method and the Schultze method yielded different contact pressure distributions. The predicted extent of the plastic region was also different according to the two mentioned methods. Thus, it is of prime significance to gauge up the predicted contact pressure distributions and the extent of plastic yielding using the two methods by means of a credible method so that a correct conclusion can be drawn, on which method prediction is more accurate.

Weighing up the results obtained by the two methods must be conducted against results obtained from an exact method. In fact, there is no rigorous method to obtain the exact contact pressure distribution. Therefore, a credible alternative must be found to pursue the sought task of passing judgment on the two methods. The finite element method has proved over the past decades its supremacy and accurate prediction for all engineering problems. As demonstrated earlier, the finite element method coupled with the elastoviscoplastic model predicted the Prandtl exact solution within an error of only 1%.

Therefore, the developed method and the Schultze method will be compared with results obtained from an elastoviscoplastic solution using the finite element mesh presented in Fig. 11. The contact pressure distribution of a rigid foundation on sandy soil whose properties is given in Table 2 was considered. The contact pressure distribution using the developed method, the Schultze method and the elastoviscoplastic finite element solution for load levels, 33.3%, 53% and 82% are given in Fig. 13, Fig.14 and Fig. 15, respectively. The results are a clear indication on the closeness of the results obtained from the developed method to that of the elastoviscoplastic finite element solution for all load levels. Moreover, the extent of the plastic region predicted by the developed method is very close to that predicted by the elastoviscoplastic finite element solution (Fig. 13 through Fig.15). The extent of the plastic region predicted by both methods is given in Table 5.

SUMMARY AND CONCLUSIONS

A new developed method for calculating the contact pressure distribution under rigid foundations was introduced. The method is based on an elastoplastic approach. First, the extent of plastic yielding zone under specific load level is determined and the contact pressure under the foundation is the combination of the contact pressure in the elastic and plastic zones. Complete derivation of the new developed method was given with necessary details.

Several examples concerning type of soils; namely φ– soil and c – φ – soil were given. Comparisons of the contact pressure distribution obtained from the developed method with that obtained from Schultze method were also presented in this work. The comparison, also, included the extent of the plastic yield zone determined by the developed method and by the Schultze method.

Evaluation of the accurateness of the developed method and the Schultze method was carried out using an elastoviscoplastic solution. The finite element solution employed advanced and precise soil modeling. The soil domain was approximated with a well developed finite element mesh comprised of 8 – noded isoparametric elements. Comparisons of the contact pressure distribution as well as the extent of plastic yield zone were made. The following conclusions are drawn from the present work:

1. The contact pressure distribution obtained by the developed method is non-linear in the elastic region as well as in the plastic region.
2. The contact pressure distribution obtained by the developed method is very close to the distribution obtained from a well developed elastoviscoplastic finite element solution.
3. The developed method predicts the extent of the plastic yield zone with high precision as compared to the elastoviscoplastic finite element solution.
4. Schultze method overpredicts the extent of plastic zone, on average, by 20%.
5. The developed method may be used by practicing engineers to determine the contact pressure distribution under rigid foundations with high degree of expected accuracy.

Nomenclature

$a = B/2$,
B = foundation width,
c = soil cohesion,
D_f = Foundation's level,
E = Modulus of elasticity,
F = the plastic yield function,

N_c, N_q and N_γ = bearing capacity parameters,

P = applied load on the foundation,
q = overburden pressure at foundation level D_f,
q_{ult} = ultimate bearing capacity,
Q = plastic potential function,
γ_1, γ_2 = unit weight above and below foundation level respectively,
γ = fluidity parameter,
υ = Poison's ratio of the soil.
ξ_1 = location of the point separating the elastic and plastic regions,
$\sigma_z(x)$ = the vertical stress at the soil-foundation interface at point x away from the center of the foundation,
$\sigma_{i,j}$ = the stresses at a point in the soil domain,
$[\dot{\varepsilon}]$ = total strain rates,
$[D]$ = elasticity matrix,
$[\sigma]$ = stresses at a particular point in the soil, and
$[\dot{\sigma}]$ = strees rates.

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